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ABSTRACT

This resource book contains 15 magic tricks based on the properties of whole numbers that are intended to spark the interest and imagination of students. Following each activity, students are asked to write about their discoveries and to create their own magic tricks. A matrix of skills for all the activities and lists of the materials required for each activity are included. Each of the activities has three sections: unraveling the trick, questions to ponder, and further investigations. The emphases of these activities are on student discovery, critical thinking, verbal and written communications, and creativity. The tricks work well in whole class or small group situations. (DDR)

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ALGEBRA MAGIC TRICKS

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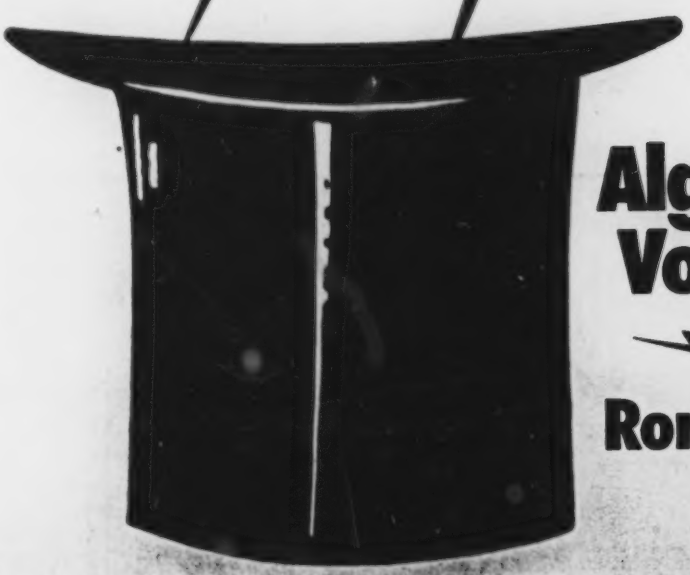

Algecadabra! Volume 2

Ronald Edwards

Critical Thinking Press & Software

SE 059336

ALGEBRA MAGIC TRICKS

Algecadabra! Volume 2

Ronald Edwards

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Introduction

Algebra Magic Tricks: Algecadabra 2 adds fifteen magic tricks to the twenty presented in *Algebra Magic Tricks: Algecadabra!* Many of the tricks are not new, for they have their origins back in the middle ages. What is new is a complete algebraic analysis of each trick. The algebra is simple enough for a student of elementary algebra to follow all the proofs provided in the text.

Each trick is presented as a mathematical exploration which includes the student observing, conjecturing, hypothesizing, verifying, and, finally, creating magic tricks.

The *TEACHER'S INSTRUCTIONS* introduce each trick and explain how to perform it. A specific example illustrates the interaction of the subject and the magician during the trick. This is followed by a detailed algebraic analysis of "why the trick works."

If the teacher wishes students to explore a trick individually or in a group, *STUDENT ACTIVITY* sheets are provided for each trick. These activity sheets lead the student through each step of the trick and provide hints and questions to be considered in the algebraic analysis. After each trick, a series of problems and questions are presented to the student as *FURTHER INVESTIGATIONS*. Hints or solutions to these investigations are presented in the *TEACHER'S INSTRUCTIONS*.

Teachers may use these materials as motivational activities to begin or end a class or as teacher-led classroom activities. Likewise, they may be presented as individual or cooperative activities. As students become more confident in their use of algebra, they can be given more freedom in the discovery process and, eventually, create their own magic tricks. At this point, students start to think algebraically.

The *MATRIX OF MATHEMATICAL CONCEPTS AND SKILLS* on the following page shows the skills necessary to understand each proof and may assist teachers in selecting activities which meet their students' mathematical development or relate to specific topics in an algebra course.

In memory of my father,
who instilled in his children
the need to know why

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Matrix of Mathematical Concepts and Skills															
Concept or Skill	**Activity**														
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Arithmetic															
basic computation	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√
expanded notation	√				√	√	√	√	√	√	√				√
multiples							√				√				√
factorization						√	√								
division theorem	√		√	√		√	√			√		√			
odd and even numbers			√												
prime numbers							√								
divisibility properties			√												√
commutative property				√		√		√				√			
distributive property		√		√		√	√	√	√			√			√
permutations	√					√									
partitioning integers			√	√								√			
Algebra															
use of variable(s)	√	√	√	√	√	√	√	√	√		√	√	√	√	√
algebraic expressions		√	√	√	√	√	√	√	√		√	√	√	√	√
polynomials				√		√		√			√	√		√	√
simplifying polynomials		√	√	√	√	√		√	√		√	√		√	√
adding polynomials		√		√		√					√				√
special products								√							
distributive property		√	√			√		√	√		√	√		√	√
factoring polynomials						√			√					√	√
solving equations					√									√	

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NUMBER 1 WHO'S GOT WHAT?

This trick dates back to the middle ages. A magician asks each person in a group to make a selection from a collection of items. The magician then predicts which person has selected each of the items. In the following version of the trick, each of three people (*P*, *Q*, and *R*) select one of three items (*a*, *b*, or *c*). For the prediction, a pile of counters consisting of 24 similar objects is used.

PERFORMING THE TRICK

1. Select three people as subjects. In your mind, label them *P*, *Q*, and *R*.
2. Place three items on a table. In your mind, label these items *a*, *b*, and *c*.
3. Turn your back to the subjects, and ask each person to select and conceal one of the items.
4. Turn to your subjects, and place 24 counters on the table; poker chips or pennies will do. Give one of these counters to *Q* and two counters to *R*. Person *P* is not given any counters.
5. Turn your back again and tell the person who selected item *a* to take as many counters as s/he presently has. Tell the person with item *b* to take three times as many counters as s/he now has and the person with item *c* to take nine times as many as s/he now has.
6. Face the subjects again, and ask one of them to do the following simple arithmetic, using a number that you suggest:
   - Divide the suggested number by 3 and state the quotient and remainder.
   - Now, divide the quotient by 3, noting the new quotient and remainder.
   - Divide this new quotient by 3 and, again, note the quotient and remainder.
7. Announce who is concealing each of the items.

HOW IT WORKS

Assign to persons *P*, *Q*, and *R* the numbers 0, 1, and 2, respectively. The first remainder identifies the person concealing item *a*, the second remainder the person concealing item *b*, and the last remainder the person with item *c*.

AN EXAMPLE

Suppose your three subjects are Anne, Anthony, and Lori and the three items you have selected are a set of keys, a pen, and a dollar bill. While your back is turned to the subjects, each selects and hides one of the items. Mentally, you assign the letters *P*, *Q*, and *R* and the numbers 0, 1, and 2, respectively, to your subjects Anne, Anthony, and Lori. Also, assign to the keys, pen, and dollar the letters *a*, *b*, and *c*, respectively.

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P = 0 = Anne

Q = 1 = Anthony

R = 2 = Lori

a = keys = same number of counters

b = pen = 3 times as many counters

c = dollar = 9 times as many counters

When you turn and face the subjects, place 24 counters on the table. Give 1 counter to Anthony (person Q), 2 to Lori (person R), but none to Anne (person P). Turn your back to the subjects and direct whoever has the keys (item a) to take as many counters from the pile as s/he now has, whoever has the pen (item b) to take three times as many counters as s/he has, and whoever has the dollar (item c) to take nine times as many as s/he has. As you turn to your subjects, inconspicuously note the number of counters left on the table. Let us suppose that there are 10 counters left. Mentally compute the number of counters that were taken by the subjects while your back was turned.

24 – (the 3 you gave out) = 21

21 – (the number left on the table) = 21 – (10) = 11

The number of counters taken, 11 in this case, is the number you will suggest as the dividend in the division process.

Ask one of the subjects to carry out the three divisions outlined in step #6 of *PERFORMING THE TRICK*. If a blackboard is available, have this done on the board. If not, have a pad of paper available.

The divisions are as follows, using your suggested number, 11:

$$\begin{array}{r} 3 \\ 3\overline{)11} \\ \underline{9} \\ 2 \end{array}$$

remainder 2
person "2" or R
(Lori) has a,
the keys

$$\begin{array}{r} 1 \\ 3\overline{)3} \\ \underline{3} \\ 0 \end{array}$$

remainder 0
person "0" or P
(Anne) has b,
the pen

$$\begin{array}{r} 0 \\ 3\overline{)1} \\ \underline{0} \\ 1 \end{array}$$

remainder 1
person "1" or Q
(Anthony) has c,
the dollar bill

Using the remainders shown above, make your predictions in a dramatic way. Note that you could have done the arithmetic to determine the three remainders yourself.

WHY IT WORKS

Since there are only six possible ways for P, Q, and R to distribute the three items a, b, and c, we can list these six outcomes and verify in each case (proof by exhaustion) that the first, second, and third remainders in the division will correctly predict who holds each item. The table below displays the six possible selections.

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Possibility	Item selected by *P*	Item selected by *Q*	Item selected by *R*
#1	*a*	*b*	*c*
#2	*a*	*c*	*b*
#3	*b*	*a*	*c*
#4	*b*	*c*	*a*
#5	*c*	*a*	*b*
#6	*c*	*b*	*a*

One of these six possibilities will be verified here. Similar verifications can be done for the other five. Consider possibility #3 in the table: *P* selects *b*, *Q* selects *a*, and *R* selects *c*. You distribute 1 counter to *Q*, 2 counters to *R*, and 0 to *P*.

When you say "whoever has item *a* take as many counters as you now have," person *Q* will take 1 counter. When you say "whoever has item *b* take three times as many counters as you now have," person *P* will take 0 counters. Finally, when you say "whoever has item *c* take nine times as many counters as you now have," person *R* will take 18 counters.

Person	Original Counters	Object Held	Counters Taken
P	0	*b*	0
Q	1	*a*	1
R	2	*c*	18

There will be 2 counters left on the table. When you note the 2 counters left and recall the fact that you gave out 3, you will know the number of counters taken: $(24 - 3) - 2 = 19$. Using 19 as the first dividend, the three divisions follow:

$$\begin{array}{r} 6 \\ 3\overline{)\,19} \\ \underline{18} \\ 1 \end{array}$$

remainder 1
person "1" or *Q*
has item *a*

$$\begin{array}{r} 2 \\ 3\overline{)\,6} \\ \underline{6} \\ 0 \end{array}$$

remainder 0
person "0" or *P*
has item *b*

$$\begin{array}{r} 0 \\ 3\overline{)\,2} \\ \underline{0} \\ 2 \end{array}$$

remainder 2
person "2" or *R*
has item *c*

These predictions prove to be true for line three in the table.

If you are familiar with the process of converting base ten numerals to base three, the three divisions above convert 19 (in base ten) to $2 \times 3^2 + 0 \times 3^1 + 1 \times 3^0$. Each term in this base three representation predicts who is holding one of the items. For example, in 2×3^2, the 2 represents the number of counters that were given to *R*. Therefore, *R* has multiplied her number by 3^2 or 9 and, thus, she chose item *c*. Similarly, the other two terms predict the items taken by *P* and *Q*.

For a discussion of converting from base ten to other bases, see *Algebra Magic Tricks: Algecadabra! Volume 1* (Critical Thinking Press & Software, 1992, p. 81).

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HINTS TO FURTHER INVESTIGATIONS

1. You would need at least $3 \times 4^3 + 2 \times 4^2 + 1 \times 4^1 + 0 \times 4^0 = 228$ counters, plus the six you give out, or a total of 234 counters.

   Try one possible distribution of the items, and compute the number *N*, representing the number of counters taken by the subjects. Using *N* as the first dividend, carry out four divisions by 4 and note the remainders. Will these remainders tell you which subject selected which item?

2. If the number *N* converts to $2 \times 4^3 + 3 \times 4^2 + 0 \times 4^1 + 1 \times 4^0$, then *Q* (person 1) has item *a*, *P* (person 0) has *b*, *S* (person 3) has *c*, and *R* (person 2) has *d*.

3. The minimum number of counters would be $4 \times 5^4 + 3 \times 5^3 + 2 \times 5^2 + 1 \times 5^1 + 0 \times 5^0 = 2930$, plus 10 counters given out, or 2940 counters. Have each person write down a number which would be equivalent to the number of counters they should have taken. If the subjects sum their numbers and carry out the five divisions by 5, you need only know the five remainders to make your predictions.

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THE TRICK: Who's Got What?

1. Select three people as subjects. Call them *P*, *Q*, and *R*.
2. Place three items on a table. Call them *a*, *b*, and *c*.
3. While your back is turned, ask each person to select one of the items and conceal it.
4. Place 24 counters (you can use poker chips) on the table. Give one chip to *Q* and two chips to *R*. Person *P* gets none.
5. With your back turned, ask whoever selected item *a* to take as many chips as s/he now has, whoever took *b* to take three times as many chips as s/he now has, and whoever has *c* to take nine times as many chips as s/he now has.
6. Face the subjects and ask one of them to carry out the following divisions with a number you suggest:
   - Divide the suggested number by 3, noting the quotient and remainder.
   - Divide the quotient by 3, noting the new quotient and remainder.
   - Divide the new quotient by 3, noting the quotient and remainder.
7. Now you can announce who is concealing each of the items.

UNRAVELING THE TRICK

1. How many ways can *P*, *Q*, and *R* make selections from the items *a*, *b*, and *c*?

2. Choose one of the possible selections found above and note the number of counters given to the subjects (3), the number taken by the subjects (call this number *N*), and the number left on the table (call this *T*). How can you determine the value of *N* if you know the value of *T*?

3. For the selection used above, carry out the three divisions (in step #6 of *THE TRICK*) using the number *N* as your first dividend. What are the three remainders?

4. Let the numbers 0, 1, and 2 be associated with the three subjects *P*, *Q*, and *R*, respectively (*P* = 0, *Q* = 1, *R* = 2). Compare your three remainders (found in #3 above) with the identities of the persons who selected items *a*, *b*, and *c*. Can you predict who took each item?

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QUESTIONS TO PONDER

1. Does the number of counters used in the trick have to be 24?

2. What is the minimum number of counters needed in the trick?

3. How many remainders are possible when a given number is divided by 3?

4. After the three divisions are carried out, will any two remainders be equal?

FURTHER INVESTIGATIONS

1. Suppose there were four subjects (*P*, *Q*, *R*, and S) selecting from four items (*a*, *b*, *c*, and *d*). Also, suppose you gave 1 counter to *Q*, 2 to *R*, 3 to S, and 0 to *P*. Determine how you would alter the trick so that you could predict which subject selected each item. (Hint: Have a subject take either the same number of counters, 4 times as many, 16 times as many, or 64 times as many as s/he has.)

2. Show how converting the number of counters taken by the subjects to a base four numeral could be used to predict who took what.

3. If there were 5 subjects, what is the minimum number of counters that would be needed? How could you alter the trick to solve the problem of needing large numbers of counters for your prediction?

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NUMBER 2 COUNTERMOVES

In the following trick, the subject seems to be in control of shifting and removing counters. Yet the magician always knows how many counters are left at the end of the trick even though the magician is not facing the subject. Use any set of similar objects as counters. The number of counters used is not important.

PERFORMING THE TRICK

1. Place a set of counters and a container on a table in front of your subject.
2. Turn your back to the table during the remainder of the trick.
3. Ask the subject to form the counters into 3 piles, each containing the same number. Have the subject arrange the piles in a row on the table. Tell the subject it is not necessary to use all of the counters provided and to put any unused ones in the container.
4. Have the subject suggest a small positive integer. Call this number *n*. Now give instructions a–e below to the subject.
   a. Move *n* (the number that the subject suggested) counters from the left pile to the middle pile.
   b. Also, move *n* counters from the right pile to the middle pile.
   c. Pick up the left pile and put it in the container.
   d. From the middle pile, remove as many counters as there are in the right pile and place them in the container.
   f. Pick up the right pile and put it in the container.
5. At this point, you announce the number of counters remaining on the table.

HOW IT WORKS

The number of counters left on the table (not counting those in the container) is always 3 times *n*, the number suggested by the subject in step #4 of the trick.

Of course, if the suggested number is more than the number of counters in each of the piles, the trick does not work. You can control this by having the subject suggest a smaller positive integer.

AN EXAMPLE

Suppose you have dumped a container of over 40 counters on a table. If people other than the subject are observing this trick, an overhead projector can be used to display the counters.

Let us suppose that the subject forms 3 piles, containing 12 counters each, and has

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returned the remaining counters to the container. When you ask for a small positive integer, suppose the subject suggests 3. This is your n value in the trick.

The table below indicates the number of counters in each pile as the subject follows the directions.

Direction #	Left pile	Middle pile	Right pile
3	12	12	12
4a	9	15	12
b	9	18	9
c	0	18	9
d	0	9	9
e	0	9	0

Since $n = 3$, $3 \times n = 9$. This is the number of counters you announce as being left on the table.

WHY IT WORKS

Suppose the subject places x counters in each pile and suggests n as the small positive integer in step #3 of the trick. Of course, you must be sure that $n \leq x$.

The table below indicates the number of counters in each pile as you go through the steps in the trick.

Step #	Left pile	Middle pile	Right pile
3	x	x	x
4a	$x - n$	$x + n$	x
b	$x - n$	$x + 2n$	$x - n$
c	0	$x + 2n$	$x - n$
d	0	$x + 2n - (x - n)$	$x - n$
e	0	$x + 2n - (x - n)$	0

Note that the number of counters left on the table is $x + 2n - (x - n) = 3n$. Thus, when you announce that there are $3n$ left, you will always be correct (as long as $n \leq x$).

HINTS TO FURTHER INVESTIGATIONS

1. One possible set of moves is as follows:
   (a) n from pile #1 to #3
   (b) n from pile #5 to #3
   (c) remove pile #1

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(d) remove from pile #3 as many counters as are in pile #5

(e) remove pile #5

(f) move pile #2 to pile #3

(g) remove from pile #3 as many counters as are in pile #4

(h) remove pile #4

Then announce the number left on the table.

2. At the end, you will find that there are 2 in pile #1 and none in pile #2. Make a table for the two piles, and follow the trick step-by-step.

3. R stands for red and G for green.

R R R R G G G G R R G R R R G R G G R R G G G R G G R R G

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THE TRICK: Countermoves

1. Select a subject and dump a container of counters (any collection of similar objects will do) on a table in front of the subject.
2. While your back is turned, have the subject form a row of three piles, each pile containing the same number of counters.
3. The subject need not use all the counters and should return any unused counters to the container.
4. Ask your subject to suggest a small positive integer. Call this *n*. While your back is still turned, give instructions a–e below to the subject.
   a. Move *n* (the number suggested by the subject) counters from the left pile to the middle pile.
   b. Move *n* counters from the right pile to the middle pile.
   c. Pick up the left pile and put it in the container.
   d. From the middle pile, remove as many counters as there are in the right pile, and place them in the container.
   e. Pick up the right pile of counters, and place it in the container.
5. At this point, you can announce the number of counters remaining.

UNRAVELING THE TRICK

1. Choose one variable to represent the number of counters in each pile. Choose another variable to represent the small positive integer suggested in step #4 of the trick.

2. Make a table indicating the number of counters in each pile after step #2 in the trick.

3. In the table, write expressions which represent the number of counters in the piles after each of the steps a–e.

4. Use the last line of your completed table to explain the trick. What would your prediction be?

QUESTIONS TO PONDER

1. What would happen in the trick if the value of your second variable were greater than that of the first variable?

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2. Does the trick work if the values of the two variables are equal?

3. Could you change the order (or alter) steps a–e in the trick?

FURTHER INVESTIGATIONS

1. Devise a COUNTERMOVE trick that starts with a row of five piles, each containing the same number of counters.

2. Analyze the following trick:

   Given 9 counters on a table, place one in your right hand (call this pile #1) and one in your left hand (call this pile #2). Now, starting with your right hand (pile #1), alternate hands in picking up the counters one at a time until there are none left on the table. With the counters concealed in your hands, begin placing the counters back on the table one at a time. When returning the counters, begin with the left hand and alternate hands until seven counters have been returned to the table. How many are left in each pile (hand)?

3. Given 15 red counters and 15 green counters, place them in a line so that when every ninth counter is discarded, the only counters remaining on the table are the 15 red ones.

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NUMBER 3 DIVINING EVEN AND ODD

In early writings, predicting the number a person thought of was called "divining" a number. In this trick, a subject is asked to think of a number and carry out a series of simple arithmetic operations. Although the subject's final result is never revealed, the magician is able to divine the subject's original number.

PERFORMING THE TRICK

1. Select a subject.
2. Ask your subject to choose any number greater than 3.
3. Ask the subject to carry out computations a–f, starting with the number chosen. If the subject's chosen number is large, a calculator may be helpful.
   a. Multiply by 3.
   b. Ask the subject, "Is your number even or odd?"

      If even, divide by 2.

      If odd, add 1 and then divide by 2.
   c. Multiply by 3.
   d. Again ask, "Is the result even or odd?"

      If even, divide by 2.

      If odd, add 1 and then divide by 2.
   e. Divide by 9, finding the quotient and remainder.
   f. Ask the subject to tell you just the quotient found in the last division.
4. Announce the number originally selected by the subject.

HOW IT WORKS

All you need to do is remember whether the subject's number was even or odd after each multiplication by 3 (questions b and d above) and know the quotient announced by the subject.

If the quotient is called Q, then the original number was

$4Q$ if the answers were even and even,

$4Q + 1$ if the answers were odd and even,

$4Q + 2$ if the answers were even and odd,

$4Q + 3$ if the answers were odd and odd.

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AN EXAMPLE

Suppose the subject picks the number 50. Then the subject computes as follows:

Step in trick	Subject's computation
2	50
3a	$3 \times 50 = 150$
b	$150 \div 2 = 75$, since 150 is even
c	$3 \times 75 = 225$
d	$225 + 1 = 226$, add 1 since 225 is odd, $226 \div 2 = 113$
e	$113 \div 9 = 12$ rem 5
f	Here the subject announces the quotient, 12. Thus, $Q = 12$.

Since the answers to the two questions were even and odd, respectively, you announce that the subject's number is $4Q + 2 = 4(12) + 2 = 50$.

WHY IT WORKS

Dividing any number by 4 results in only four possible remainders: 0, 1, 2, and 3. Thus, any number, N, can be expressed in one and only one of the following forms: (1) $4Q$, (2) $4Q + 1$, (3) $4Q + 2$, or (4) $4Q + 3$, where Q is an integer. Subdividing a set in this fashion is referred to as partitioning the set (here using division by 4).

We can now show that the prediction will work for each of the four possible cases.

In case 1, the subject's computations are $4Q$

$3(4Q) = 12Q$ (even)

$12Q \div 2 = 6Q$

$3(6Q) = 18Q$ (even)

$18Q \div 2 = 9Q$

$9Q \div 9 = Q$ (the quotient, Q, is announced)

Thus, for the answers even and even, the subject's number is $4Q$.

The computations for case 2 are $4Q + 1$

$3(4Q + 1) = 12Q + 3$ (odd)

$(12Q + 3) + 1 = 12Q + 4$

$(12Q + 4) \div 2 = 6Q + 2$

$3(6Q + 2) = 18Q + 6$ (even)

$(18Q + 6) \div 2 = 9Q + 3$

$(9Q + 3) \div 9 = Q$ rem 3 (the quotient, Q, is announced)

Thus, for the answers odd and even, the subject's number is $4Q + 1$.

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In case 3, the subject's computations are $4Q + 2$

$3(4Q + 2) = 12Q + 6$ (even)

$(12Q + 6) \div 2 = 6Q + 3$

$3(6Q + 3) = 18Q + 9$ (odd)

$(18Q + 9) + 1 = 18Q + 10$

$(18Q + 10) \div 2 = 9Q + 5$

$(9Q + 5) \div 9 = Q$ rem 5 (the quotient, Q, is announced)

Thus, for the answers even and odd, the subject's number is $4Q. + 2$

The computations for case 4 are $4Q + 3$

$3(4Q + 3) = 12Q + 9$ (odd)

$(12Q + 9) + 1 = 12Q + 10$

$(12Q + 10) \div 2 = 6Q + 5$

$3(6Q + 5) = 18Q + 15$ (odd)

$18Q + 15 + 1 = 18Q + 16$

$(18Q + 16) \div 2 = 9Q + 8$

$(9Q + 8) \div 9 = Q$ rem 8 (the quotient, Q, is announced)

Thus, for the answers odd and odd, the subject's number is $4Q + 3$.

HINTS TO FURTHER INVESTIGATIONS

1. In each case, the quotient is zero.
2. Note that any number can be expressed in one and only one of the forms $2Q$ or $2Q + 1$. Test each possibility.
3. Since you can predict the subject's number after the quotient is announced, you can add any steps to alter this number. Then you will be able to predict not only the original number thought of by the subject but also the number resulting from the added instructions.

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THE TRICK: Divining Even and Odd

1. Select a subject.
2. Ask your subject to choose any number greater than 3.
3. Ask the subject to carry out the following computations with the number chosen:
   a. Multiply by 3.
   b. Ask if the number is even or odd.
      If even, divide by 2. If odd, add 1 and then divide by 2.
   c. Multiply by 3.
   d. Ask if the result is even or odd.
      If even, divide by 2. If odd, add 1 and then divide by 2.
   e. Divide by 9, finding the quotient and remainder.
   f. Have the subject announce the quotient found in the division by 9.
4. At this point, you will be able to announce the number originally selected by the subject.

UNRAVELING THE TRICK

1. What possible remainders are there when a number is divided by 4?

2. If N represents the number chosen by the subject, explain why N can be expressed in one and only one of the following forms: $4Q$, $4Q + 1$, $4Q + 2$, or $4Q + 3$, where Q is an integer.

3. Suppose the number chosen is of the form $4Q + 1$. Follow steps a–f in the trick and determine the quotient announced and the answers to the questions in b and d of the trick.

4. Examine the forms $4Q$, $4Q + 2$, and $4Q + 3$ as was done for $4Q + 1$ in #3 above.

5. Use your observations in #3 and #4 above to explain how you can predict the subject's number.

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QUESTIONS TO PONDER

1. What happens in the trick if the subject's selected number is 1, 2, or 3?

2. Why isn't the remainder (in the final division) needed in the prediction?

3. Could you predict the subject's number if only the remainder were known?

4. Could you predict the subject's number if you noted that any number can be written in one and only one of the forms $3Q$, $3Q + 1$, or $3Q + 2$?

FURTHER INVESTIGATIONS

1. Why were the numbers 1, 2, and 3 excluded as possible choices in the trick?

2. Analyze the following version of the trick:
   Have your subject think of a number.
   Multiply by 3.
   Ask if the result is even or odd.
   If even, divide by 2. If odd, add 1 and divide by 2.
   Multiply by 3.
   Divide by 9, announcing only the integer quotient.
   The prediction is $2Q$ if the answer was even and $2Q + 1$ if the answer was odd.

3. How could you alter the original trick to arrive at a different prediction?

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NUMBER 4 DIVINING SUMS

The following is a simple arithmetic trick which was first played on me by a young friend. A subject is allowed to make a selection of numbers from a table, but before the subject can sum the chosen numbers, the magician knows the result. Thus, the magician can reveal the sum in some dramatic way.

PERFORMING THE TRICK

1. Present your subject with the table pictured at the right. (See p. 22 for reproducible table.) Now, have the subject follow your instructions, out of your sight.
2. Select any number from row #1 in the table. Cross out all other numbers that are in the same column as the number chosen.
3. Select any number from row #2 (exclude any crossed-out number). Now, cross out all other numbers that are in the same column as the number chosen.
4. Repeat this process for subsequent rows until no more selections can be made.
5. Sum the numbers you have chosen.
6. The sum is known before the subject selects the first number and can be revealed at any time.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16
17	18	19	20

HOW IT WORKS

The sum is always 34.

Before the trick begins, this prediction can be sealed in an envelope and handed to the subject with the table of values. When the subject determines the sum, the envelope can be opened to reveal your prediction. This trick can be quickly repeated any number of times by having the subject key the choices directly on a calculator.

AN EXAMPLE

Suppose the subject's choices are indicated by the circled numbers on the table at right.

The sum of these choices is 3 + 8 + 9 + 14 = 34.

~~1~~	~~2~~	(3)	~~4~~
~~5~~	~~6~~	~~7~~	(8)
(9)	~~10~~	~~11~~	~~12~~
~~13~~	(14)	~~15~~	~~16~~
~~17~~	~~18~~	~~19~~	20

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WHY IT WORKS

Note that the four columns in the table partition the numbers from 1 to 20 into four sets. The numbers in column #1 are of the form $4K + 1$, where $K = 0, 1, 2, 3$. The numbers in column #2 are of the form $4K + 2$, where $K = 0, 1, 2, 3$. The numbers in column #3 are of the form $4K + 3$ and in column #4 of the form $4K + 4$, where $K = 0, 1, 2, 3$.

All of the numbers ($4K + 1$, $4K + 2$, $4K + 3$, and $4K + 4$) in row #1 have a K value of 0, all numbers in row #2 have a K value of 1, in row #3 a value of 2, and in row #4 a value of 3.

Also, note that the subject never gets to select a number from row #5 since all numbers in the table will have been chosen or crossed out by then.

One key to predicting the sum is that no two numbers selected will be of the same form since they are each from a different column. The other key to the prediction is that no two forms will have the same K value, since they are from different rows.

When the sum is computed, it will appear as follows:

$$\begin{aligned} S &= (4K_1 + 1) + (4K_2 + 2) + (4K_3 + 3) + (4K_4 + 4) \\ &= (4K_1 + 4K_2 + 4K_3 + 4K_4) + (1 + 2 + 3 + 4) \\ &= 4(K_1 + K_2 + K_3 + K_4) + 10 \quad \text{but no two } K_i\text{s are equal, so the sum of the } K_i\text{s is } 0 + 1 + 2 + 3 = 6 \\ &= 4(6) + 10 = 34. \end{aligned}$$

The trick can also be verified using *proof by exhaustion*, that is, listing all possible outcomes to the trick and verifying each arithmetically. The number of choices in rows 1–4 are 4, 3, 2, and 1, respectively. Thus, there are $4 \times 3 \times 2 \times 1$ or 24 possible choices for the subject. With the aid of a calculator, the predicted sum for each of the 24 choices can be verified quickly.

HINTS TO FURTHER INVESTIGATIONS

1. $S = 5(0 + 1 + 2 + 3 + 4) + (1 + 2 + 3 + 4 + 5) = 5(10) + 15 = 65.$

1	**2**	**3**	**4**	**5**
6	**7**	**8**	**9**	**10**
11	**12**	**13**	**14**	**15**
16	**17**	**18**	**19**	**20**
21	**22**	**23**	**24**	**25**
26	**27**	**28**	**29**	**30**

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2. $S = 10(0 + 1 + \ldots + 9) + (1 + 2 + \ldots + 10) = 10(45) + 55 = 505.$

   The table is arranged with ten rows and ten columns.

3. Use the formula $1 + 2 + \ldots + x = \frac{x(x+1)}{2}$

   $S = n(0 + 1 + 2 + \ldots + (n-1)) + (1 + 2 + \ldots + n)$

   $S = \frac{n((n-1)(n-1+1))}{2} + \frac{n(n+1)}{2}$

   $S = \frac{n((n-1)n)}{2} + \frac{n(n+1)}{2}$

   $S = \frac{n(n^2-n)}{2} + \frac{n(n+1)}{2}$

   $S = \frac{n}{2}(n^2 - n + n + 1)$

   $S = \frac{n}{2}(n^2 + 1)$

   Thus, for $n = 10$, $S = \frac{10}{2}(10^2 + 1) = 5\ (101) = 505.$

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THE TRICK: Divining Sums

1. Present a subject with the table pictured at the right. Have the subject follow instructions #2–5 out of your sight.
2. Select any number from row #1 in the table. Cross out all other numbers that are in the same column as the number chosen.
3. Select any number from row #2 (exclude any crossed-out number). Now cross out all other numbers that are in the same column as the number chosen.
4. Repeat this process for subsequent rows, until no more selections can be made.
5. Sum the numbers chosen.
6. Before the subject selects his/her first number from the table, the sum is already known.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16
17	18	19	20

UNRAVELING THE TRICK

1. Using partitioning with division by 4, what is the form of all numbers in column #1? In column #2? In column #3? In column #4? (Refer to partitioning by 4 in TRICK #3.)

2. Will the subject ever select a number from row #5?

3. If the numbers in row #3 were expressed in the forms $4K + 1$, $4K + 2$, $4K + 3$, and $4K + 4$, what would be the value of K? What is the K value in row #1? In row #2? In row #4?

4. Are any two chosen numbers in the same row? In the same column?

5. In light of the answers to #1–4 above, answer the following questions:

   Will each chosen number have a different form? Can any of the K values for the chosen numbers be the same?

6. Write an expression for the sum of the four chosen numbers. Use your discoveries from #1–5 above to determine the sum of the four chosen numbers.

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QUESTIONS TO PONDER

1. Why are the numbers in the table arranged in 4 columns?

2. How many numbers will the subject get to select in the trick?

3. Why are there 5 rows in the table?

4. Could partitioning be done using division by a number other than 4?

FURTHER INVESTIGATIONS

1. Extend the trick by listing the numbers 1–30 in a table. Arrange the numbers in the table using a partition with division by 5. What would your prediction be for the sum of selected numbers from this table?

2. Extend the trick for the numbers 1–100. What is your prediction?

3. If a table of counting numbers were produced using division by n (where n is greater than 2), find a formula for predicting the sum of the numbers selected.

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1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16
17	18	19	20

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NUMBER 5 DIVINING TABULAR SUMS

This is another divining trick using a table of values.

PERFORMING THE TRICK

1. Give your subject the table of numbers pictured at the right. (See p. 27 for a reproducible table.)
2. Ask your subject to choose one number from each of the columns in the table.
3. Have the subject write these numbers in a column on a blackboard or on a sheet of paper.
4. Now, direct the subject to sum the column of figures.
5. Before the subject begins the sum, you will know the result. The answer can be revealed in any way you see fit.

462	**781**	**671**	**96**
165	**286**	**275**	**294**
264	**484**	**374**	**591**
363	**583**	**572**	**393**

HOW IT WORKS

The sum of the chosen numbers is either a three- or four-digit number.

The last two digits of the sum are the sum of the unit's digits of the four chosen numbers. Call this number *U*. If *U* is a single digit, prefix a zero. The first two digits of the sum are found by subtracting *U* from 30. If this difference is a single digit, the sum will be a three-digit number.

AN EXAMPLE

Suppose the subject chooses the following numbers: 264, 583, 374, 96.

While the subject is writing these on the board, you mentally compute the sum of the unit's column (4 + 3 + 4 + 6 = 17), and then subtract this sum from 30 (30 – 17 = 13). Thus, your prediction is 1317.

```
 264
 583
 374
  96
```

You can write 1317 on a card and place it face down on a table as you direct the subject to sum the column of figures.

When the sum is complete, have the subject pick up your card and check the prediction. They are the same. This trick can be repeated any number of times, and you can produce the sum as soon as the last chosen number is written on the board.

WHY IT WORKS

The table of numbers used in the trick is constructed in a very special way.

All numbers in column #1 have a 6 in the ten's place. Those in columns #2, #3, and #4 have 8s, 7s, and 9s in the ten's place, respectively.

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Since each of the four selected numbers is chosen from a different column, the sums of the ten's digits of these numbers is 60 + 80 + 70 + 90 = 300. Note the zeros in the ten's place and unit's place of this sum.

Let U = the sum of the unit's column of the chosen numbers. U is always a one- or two-digit number. If U has two digits, then U will represent the last two digits in the sum. If U is one digit, prefix a 0 to U.

You will also note that the sum of the digits of each number in column #1 is 12. The sums of the digits of each number in columns #2, #3, and #4 are 16, 14, and 15, respectively.

Thus, the sum of the digits in the hundred's column (H) for the four chosen numbers can be computed as follows:

H = (sum of all digits) – (sum of digits in the ten's columns) – (sum of digits in the unit's columns)

$= (12 + 16 + 14 + 15) - (30) - (U)$

$= 57 - 30 - U$

$= 27 - U.$

But, since a 3 will be carried from the 30 tens, the hundreds will be $H + 3 = (27 - U) + 3 = 30 - U$. Thus, we can predict the first two digits in the final sum by subtracting U from 30. It is possible that H is a single digit, making the sum a three-digit number.

HINTS TO FURTHER INVESTIGATIONS

1. At right is one possibility.

393	**285**	**671**	**166**
492	**384**	**572**	**463**
195	**285**	**473**	**265**
96	**681**	**176**	**364**

2. The last two digits are the sum of the unit's column of the five chosen numbers. The first two digits are $(50 - U)$.
3. Below is a possibility. Its prediction rules are as follows:
   - The last two digits are the sum of the unit's column, U.
   - The first two digits are $(40 - U)$.

630	**165**	**343**	**720**	**383**	**175**
432	**462**	**244**	**423**	**581**	**571**
234	**660**	**145**	**324**	**185**	**373**
135	**363**	**541**	**621**	**482**	**670**
531	**264**	**442**	**225**	**680**	**274**
333	**561**	**640**	**522**	**284**	**472**

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THE TRICK: Divining Sums

1. Choose a subject and give him or her a copy of the table pictured to the right.
2. Ask the subject to choose one number from each of the columns in the table.
3. Have the subject write these numbers in a column on a blackboard or on a sheet of paper.
4. Direct the subject to sum the column of figures.
5. Before the subject begins the sum, you will know the result.

462	**781**	**671**	**96**
165	**286**	**275**	**294**
264	**484**	**374**	**591**
363	**583**	**572**	**393**

UNRAVELING THE TRICK

1. What do all the numbers in column #1 have in common? In column #2? In column #3? In column #4?

2. If you selected one number from each column, what is the sum of all digits in the ten's places of the chosen numbers?

3. How could you determine the last two digits (unit's place and ten's place) in the sum of the four chosen numbers?

4. What is the sum of the three digits of each number appearing in column #1? Column #2? Column #3? Column #4?

5. What is the sum of all the digits in the four chosen numbers?

6. Show that the first two digits in the final sum can be determined by subtracting the sum of the unit's digits of the four chosen numbers from 30.

QUESTIONS TO PONDER

1. If the sum of the unit's digits of the chosen numbers is a single digit, how does this affect your prediction of the last two digits in the sum?

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2. What is the maximum value of the sum of the unit's digits of the selected numbers?

3. Is the computed sum always a four-digit number?

4. Could the ten's places of the numbers in the table be changed and still allow you to predict the final sum easily?

FURTHER INVESTIGATIONS

1. Construct another 4×4 table of values that uses the same prediction rules.

2. The magician Royal Vale Heath proposed the following 5×5 table. Using his table in the trick, how would you predict the sum of the five numbers chosen by your subject?

69	**345**	**186**	**872**	**756**
366	**642**	**582**	**278**	**558**
168	**246**	**87**	**575**	**657**
762	**147**	**285**	**377**	**954**
663	**543**	**483**	**179**	**855**
564	**48**	**384**	**674**	**459**

3. Can you produce a 6×6 table for the trick? What would your prediction rules be for the table?

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462	781	671	96
165	286	275	294
264	484	374	591
363	583	572	393

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NUMBER 6 DIVINING ROD

In earlier times, a divining rod was used to locate underground water. In this trick, with the aid of your subject, you literally divine "ROD." The trick uses the set of permutations of a number selected by your subject.

PERFORMING THE TRICK

1. Ask your subject to choose any 3-digit number consisting of 3 different digits. Then have your subject carry out instructions a–e below.
   a. Make a list of all the permutations of the 3-digit number. (There should be six.)
   b. Sum the list of permutations.
   c. Divide the sum by 3.
   d. Divide the resulting quotient by 2.
   e. Divide the resulting quotient by the sum of the digits of the original 3-digit number.
2. Before the subject has selected the original 3-digit number, you know the final result.

HOW IT WORKS

The result is always 37, and here is where ROD comes in. The letter *R* is the 18th letter in the alphabet, *O* is the 15th letter, and *D* is the 4th. So ROD converts to 18 + 15 + 4 = 37.

Before beginning the trick, seal in an envelope a sheet of paper on which the word ROD is written. Have the subject hold the envelope during the trick. At the end of the computation, have the subject compare your prediction with the number determined in the trick. Your subject or any spectators will probably be confused until your convert ROD to the number 37.

AN EXAMPLE

If the number selected was 457, then the six permutations are as follows:

457	547	745
475	574	754

The sum of the six numbers (457 + 475 + 547 + 574 + 745 + 754) is 3552 and the results of the three divisions are as follows:

3552 ÷ 3 = 1184

1184 ÷ 2 = 592

The sum of the original digits is 16, and 592 ÷ 16 = 37, which is your prediction.

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WHY IT WORKS

Let the digits in the chosen number be a, b, and c. Note that no two of these variables have the same value. Thus, there are six permutations.

$a\,b\,c$

$a\,c\,b$

$b\,a\,c$

$b\,c\,a$

$c\,a\,b$

$c\,b\,a$

The sums for the three columns are shown below.

hundred's column	ten's column	unit's column
$2a + 2b + 2c$	$2a + 2b + 2c$	$2a + 2b + 2c$

Thus, the sum, S, expressed in expanded notation is

$$S = (2a + 2b + 2c)100 + (2a + 2b + 2c)10 + (2a + 2b + 2c)$$
$$= 200(a + b + c) + 20(a + b + c) + 2(a + b + c)$$
$$= 222(a + b + c).$$

The result of the three divisions yields the following values:

$$S \div 3 = 222(a + b + c) \div 3 = 74(a + b + c)$$

$$74(a + b + c) \div 2 = 37(a + b + c)$$

$$37(a + b + c) \div (a + b + c) = 37.$$

Hence, the result is always 37.

HINTS TO FURTHER INVESTIGATIONS

1. Since there are $9 \times 9 \times 8 = 648$ possible choices for the original 3-digit number, it depends on how ambitious you are.
2. It works, if you consider the permutations with leading zeros to be 2-digit numbers. Try it with the set of permutations 250, 205, 025, 052, 520, and 502.
3. If two digits match, there are only three permutations, and the sum may not be divisible by 2. Try dividing the sum by 3 and the result by the sum of the digits. If all three digits are identical, how many permutations are there?
4. Start with a 4-digit number that has four distinct digits: a, b, c, and d. You should find 24 permutations and $S = 6666(a + b + c + d)$. Look for a series of arithmetic computations to convert S into 37.

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THE TRICK: Divining Rod

1. Ask a subject to choose a 3-digit number consisting of 3 different digits; then, have the subject carry out instructions #2–6 below.
   a. Make a list of all permutations of the 3-digit number. (There should be six.)
   b. Sum the list of permutations.
   c. Divide the sum by 3.
   d. Divide the resulting quotient by 2.
   e. Divide the resulting quotient by the sum of the digits of the original number.
2. Before the arithmetic is started, you can predict the outcome.

UNRAVELING THE TRICK

1. Select three variables to represent the hundred's, ten's, and unit's digits of the chosen number.

2. Write the six possible permutations of the chosen number in a column.

3. Find the sum of the six permutations (separately sum the hundred's, ten's, and unit's digits), and express the sum in expanded notation.

4. Starting with the algebraic expression for the sum, carry out the three divisions outlined in the trick.

5. Use the results of the last step to explain the trick and determine the prediction.

QUESTIONS TO PONDER

1. What restrictions are there on the three variables used?

2. In the chosen number, could two of the digits be identical? Could three be identical?

3. Is the order of the three divisions important to the outcome?

4. In choosing a 3-digit number with 3 different digits, how many choices does the subject have?

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FURTHER INVESTIGATIONS

1. Is it feasible to verify this trick through proof by exhaustion, that is, listing all possible choices and verifying each separately?

2. Examine what happens if either the ten's digit or unit's digit of the chosen number is zero. Does the trick work?

3. Try modifying the trick so the subject can select a 3-digit number where exactly two of the digits are identical. What would happen if three of the digits were identical?

4. Develop a trick that begins with the subject selecting a 4-digit number, listing its permutations, and summing them.

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NUMBER 7 COMPUTING IN CYCLES

A set of cyclic numbers can be formed by shifting each digit in a given number one place to the left and moving the first digit to the rightmost place. If the shifting process continues until all possible sequences are formed, you create a cyclic set. Thus, a cyclic set formed from 2468 includes 4682, 6824, 8246, and 2468, itself. This trick uses the cyclic set formed from the number 142857. As you lead your subject through a series of simple computations, you can predict many intriguing properties of this magic number.

PERFORMING FOUR TRICKS

(A) 1. Ask your subject to write the number 142857 on a sheet of paper and cross out one of the digits.

2. Ask the subject to sum the remaining digits. If the sum is a 2-digit number, repeat the summing until a single-digit number is formed.

3. As soon as your subject announces the single-digit sum, you can identify the deleted digit.

(B) 1. Ask your subject to select any single digit (1–9).

2. As soon as you are given the digit, you can announce a prediction and have the subject write it down.

3. Now, ask your subject to multiply 142857 by the selected digit. Then compare the product to your prediction. They match.

(C) 1. Explain how cyclic numbers are formed, and have your subject write the six cyclic numbers for 142857.

2. Have your subject sum the six numbers. A calculator would be helpful.

3. While this is being computed, you can display your prediction in any way that seems suitable.

(D) 1. Have your subject use a calculator to carry out the following four divisions, starting with the number 142857 as dividend:

- Divide by 3.
- Divide the quotient by 11.
- Divide this quotient by 9.
- Divide this quotient by 13.

2. Before starting this trick, your prediction can be sealed in an envelope.

HOW THEY WORK

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(A) The deleted digit will always be 9 minus the digit that is announced by the subject.

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(B) The products when multiplying by the digits 2 through 6 are the members of the cyclic set

$142857 \times 2 = 285714$

$142857 \times 3 = 428571$

$142857 \times 4 = 571428$

$142857 \times 5 = 714285$

$142857 \times 6 = 857142$

Note that the multiplier times the unit's digit of 142857 tells you the last digit in the product and, hence, you will know the product since it is in the cyclic set.

The products for 7, 8, and 9 are

$142857 \times 7 = 999999$

$142857 \times 8 = 1142856$ (the first plus the last digit is 7)

$142857 \times 9 = 1285713$ (the first plus the last digit is 4)

(C) The sum is 2999997.

(D) The answer is 37 (which gives another way of divining "ROD"; see TRICK #5).

WHY THEY WORK

(A) The sum of the digits of 142857 is 27. Thus, when a digit, D, is deleted, the sum will be 27 – D. When the digits in the sum are added, the number announced will be 9 – *D*. If you subtract the number announced from 9, the result is

$$9 - (9 - D) = 9 - 9 + D = D, \text{ the deleted digit.}$$

(B) The answers can be verified by multiplication, but the reason these products occur is because of the digits in the repeating decimal for 1/7. As a repeating decimal, $1/7 = 0.\overline{142857}$, where the block of digits "142857" repeats infinitely.

Since, $7\overline{)1.00}$ (.14; 7; 30; 28; 2), we have $\frac{1}{7} = .14\frac{2}{7}$

Thus, in the mixed decimal, $.14\frac{2}{7}$, the 2/7 yields the repeating digits .285714. A similar argument can be provided for multiplication of 142857 by 3, 4, 5, and 6. When multiplying by 7, you have $1 = 7 \times (1/7) = 7 \times .142857 = 0.\overline{999999}$.

(C) The sum of the digits in 142857 is 27. When summing the six cyclic numbers, each place-value column will have a sum of 27. Hence, this sum in expanded notation will be

$S = 27(100000) + 27(10000) + 27(1000) + 27(100) + 27(10) + 27$

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= 27(111111), which gives you

$$\begin{array}{r} 27 \\ \times 111111 \\ \hline 27 \\ 27 \\ 27 \\ 27 \\ 27 \\ 27 \\ \hline 2999997 \end{array}$$

(D) The number 142857 factors as follows: $142857 = 3 \times 9 \times 11 \times 13 \times 37$.

Thus, the four divisions suggested in the trick produce the following result:

$$\frac{\cancel{3} \times \cancel{9} \times \cancel{11} \times \cancel{13} \times 37}{\cancel{3} \times \cancel{11} \times \cancel{9} \times \cancel{13}} = 37$$

HINTS TO FURTHER INVESTIGATIONS

1. The products are

$$\begin{aligned}
0{,}588{,}235{,}294{,}117{,}647 \times 2 &= 1{,}176{,}470{,}588{,}235{,}294 \\
0{,}588{,}235{,}294{,}117{,}647 \times 3 &= 1{,}764{,}705{,}882{,}352{,}941 \\
0{,}588{,}235{,}294{,}117{,}647 \times 4 &= 2{,}352{,}941{,}176{,}470{,}588 \\
0{,}588{,}235{,}294{,}117{,}647 \times 5 &= 2{,}941{,}176{,}470{,}588{,}235 \\
0{,}588{,}235{,}294{,}117{,}647 \times 6 &= 3{,}529{,}411{,}764{,}705{,}882 \\
0{,}588{,}235{,}294{,}117{,}647 \times 7 &= 4{,}117{,}647{,}058{,}823{,}529 \\
0{,}588{,}235{,}294{,}117{,}647 \times 8 &= 4{,}705{,}882{,}352{,}941{,}176 \\
0{,}588{,}235{,}294{,}117{,}647 \times 9 &= 5{,}294{,}117{,}647{,}058{,}823 \\
0{,}588{,}235{,}294{,}117{,}647 \times 10 &= 5{,}882{,}352{,}941{,}176{,}470 \\
0{,}588{,}235{,}294{,}117{,}647 \times 11 &= 6{,}470{,}588{,}235{,}294{,}117 \\
0{,}588{,}235{,}294{,}117{,}647 \times 12 &= 7{,}058{,}823{,}529{,}411{,}764 \\
0{,}588{,}235{,}294{,}117{,}647 \times 13 &= 7{,}647{,}058{,}823{,}529{,}411 \\
0{,}588{,}235{,}294{,}117{,}647 \times 14 &= 8{,}235{,}294{,}117{,}647{,}058 \\
0{,}588{,}235{,}294{,}117{,}647 \times 15 &= 8{,}823{,}529{,}411{,}764{,}705 \\
0{,}588{,}235{,}294{,}117{,}647 \times 16 &= 9{,}411{,}764{,}705{,}882{,}352 \\
0{,}588{,}235{,}294{,}117{,}647 \times 17 &= 9{,}999{,}999{,}999{,}999{,}999
\end{aligned}$$

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2. For 142857, $S = (1+4+2+8+5+7)\frac{(1000000-1)}{10-1} = 27\frac{(999999)}{9}$

$= 27(111111)$

$= 2{,}999{,}997$

For 0588235294117647, $S = (5+8+8+\ldots+4+7)\frac{(10^{16}-1)}{10-1}$

$= 72\frac{(999\,999\,999\,999\,999)}{9}$

$= 72(111\,111\,111\,111\,111)$

$= 7\,999\,999\,999\,999\,999\,2.$

3. As an example, consider the following:

$$\begin{array}{r} 142857 \\ \times 37416 \\ \hline 857142 \\ 142857 \\ 571428 \\ 999999 \\ 428571 \\ \hline 5345137512 \end{array}$$

$5345\ \ 137512 \rightarrow \begin{array}{r} 137512 \\ +\ 5345 \\ \hline 142857 \end{array}$

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THE TRICK: Computing in Cycles

Computing in Cycles includes four computational tricks (A–D below) that use the magic number 142857.

(A) 1. Ask a subject to write the number 142857 on a sheet of paper and cross out any one of the digits.

2. Ask the subject to sum the remaining digits. If the sum is a 2-digit number, repeat summing the digits until there is a single digit.

3. As soon as the subject announces this single-digit sum, you can identify the digit that was deleted.

(B) 1. Ask your subject to select any single digit (1–9).

2. As soon as you are told the digit, you will give the subject a prediction to write on the paper.

3. Now, ask the subject to multiply 142857 by the digit selected. When the subject compares this product with your prediction, they will always match.

(C) 1. Have the subject generate the six cyclic numbers formed from 142857.

2. Using a calculator, have the subject sum the six numbers.

3. While this is being computed, you can display your prediction.

(D) 1. Using the calculator, have the subject carry out the following four divisions, starting with 142857 as dividend:

- Divide by 3.
- Divide the quotient by 11.
- Divide this quotient 9.
- Divide this quotient by 13.

2. Before starting the trick, your prediction can be sealed in an envelope.

UNRAVELING THE TRICK

(A) 1. What is the sum of the digits in 142857?

2. Pair the digits in 142857 so that the sum of each pair is the same. If *D* represents one digit, how would you express the digit paired with it?

3. What is the sum of the digits when *D* is deleted from the number? Use this observation to predict the deleted digit.

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(B) 1. Multiply 142857 by each of the digits 1 through 6. What do you notice about each of the products?

2. How could you remember these products?

3. Multiply 142857 by each of the digits 7, 8, and 9. What do you notice about the products?

4. How could you remember these three products?

(C) 1. Write the six cyclic numbers in a column. What is the sum of the digits in each place-value column?

2. Express the sum of the cyclic numbers in expanded notation, and then factor out the sum of the digits in each place-value column.

3. Use the result in step #2 to predict the sum of the six cyclic numbers.

(D) 1. Factor 142857 into its prime factors.

2. Carry out the four divisions listed in the trick on the factored form of 142857.

3. What is your prediction? Is this a familiar number?

QUESTIONS TO PONDER

(A) 1. Does this trick work for any given number?

2. Can you relate this trick to the process of "casting out nines"? (See *Algebra Magic Tricks: Algecadabra!* p. 54.)

3. Will this prediction trick work with any number in which the sum of the digits is a multiple of 9?

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(B) 1. When 1/7 is converted to a repeating decimal, what are the digits that repeat?

2. Show that 1/7 can be expressed as the mixed decimal $0.1\frac{3}{7}$. How does this help to explain why $3 \times 142857 = 428571$?

3. Does this trick work for any cyclic set?

(C) 1. Could this technique be used to sum any set of cyclic numbers?

(D) 1. Is any other number in the cyclic set divisible by 37?

2. Can you make up a 6-digit number for which no two digits are the same and four successive divisions will lead to 37?

FURTHER INVESTIGATIONS

1. If p is a prime number and the decimal expansion has $p - 1$ repeating digits (as is the case for 142857), then the cyclic set can be produced from the repeating block of digits by multiplication. The decimal expansion of 1/17 is .0588235294117647. Show that 0588235294117647 can be used to generate a cyclic set. (Note that the leading zero cycles through this set.)

2. A formula for the sum of a cyclic set, S, is as follows:

   S = (the sum of the digits in any place-value column)$\frac{(10^n - 1)}{10 - 1}$

   where n = the number of elements in the set. Compute S for 142857 and for 0588235294117647.

3. Try the following:

   (1) Multiply 142857 by a large number (say five or more digits).
   (2) Split up the product into groups of six digits starting from the right.
   (3) Sum these groups.
   (4) If the sum is more than six digits, repeat steps #2–4.
   (5) What do you observe about this sum?

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NUMBER 8 COMPUTING COMPLIMENTS

You have, no doubt, heard about people who have a phenomenal ability to carry out the most complex computations with lightning speed. The next two tricks will make you a "lightning calculator" in a limited way.

PERFORMING THE TRICK

1. Have someone suggest two 2-digit (or 3-digit) numbers. The trick will work best if both numbers are slightly below 100 (or 1000 for 3-digit numbers) or slightly above 100 (or 1000 for 3-digit numbers).
2. While someone writes these two numbers in a column, you will immediately write the product of the two numbers. With the aid of a calculator, someone can verify your solution.

HOW IT WORKS

To perform this trick, you must know what is meant by the *complement* and the *supplement* of a number.

- The complement is the difference between the number and the next highest power of 10. Thus, the complement of 94 is 100 – 94 = 6, and the complement of 978 is 1000 – 978 = 22.
- The supplement is the difference between the number and the next smallest power of 10. The supplement of 109 is 109 – 100 = 9, and the supplement of 1012 is 1012 – 1000 = 12.

Now consider two cases:

(A) the two numbers are close to, but less than, 100 (or 1000 for 3-digit numbers); and

(B) the two numbers are close to, but larger than, 100 (or 1000 for 3-digit numbers).

Case (A): If the numbers are both 2-digits, then the product will have four digits. The first two digits are one of the numbers minus the complement of the other. The last two digits are the product of the two complements. Note that you must prefix a 0 if the product is a single digit.

If the numbers are both 3-digits, the product has six digits. The first three digits are one number minus the complement of the other. The last three digits are the product of the complements, with 0s prefixed if necessary.

Case (B): If the numbers are 3-digits, the product will have five digits. The first three are one number plus the supplement of the other. The last two are the product of the supplements with a prefixed 0 if necessary.

If the numbers have four digits, then the product will have seven digits. The first four are

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one number plus the supplement of the other. The last three are the product of the supplements. If this is less than three digits, prefix 0s.

AN EXAMPLE

Case (A)	given numbers	complements	first digits	last digits
1.	96	4	96 – 6 = 90	4 × 6 = 24
	× 94	6	(or 94 – 4)	

Thus, 96 × 94 = 9024

2.	998	2	988 – 2 = 986	12 × 2 = 024
	× 988	12	(or 998 – 12)	

Thus, 998 × 988 = 986024

Case (B)	given numbers	supplements	first digits	last digits
1.	107	7	112 + 7 = 119	12 × 7 = 84
	× 112	12	(or 107 + 12)	

Thus, 107 × 112 = 11984

2.	1009	9	1009 + 15 = 1024	15 × 9 = 135
	× 1015	15	(or 1015 + 9)	

Thus, 1009 × 1015 = 1024135

3.	108	8	108 + 17 = 125	17 × 8 = 136
	× 117	17	(or 117 + 8)	

Thus, 108 × 117 = 12636*

* It is possible that the product of the complements contains an extra digit. In the product 108 × 117, the rule states that the first three digits are 125 (or 117 + 8) and the last two digits are 136 (or 17 × 8). Here the 1 in the hundred's place must be added to 125, making the product 12636.

WHY IT WORKS

Case (A): Suppose you are multiplying two 2-digit numbers N and M. The complement of N is $100 - N$, and the complement of M is $100 - M$.

The beginning digits in the product may be expressed as $N - (100 - M)$ and the last digits by $(100 - N)(100 - M)$. In expanded notation this number is

$$100(N - (100 - M)) + (100 - N)(100 - M)$$

$$= 100N - 10000 + 100M + 10000 - 100N - 100M + MN$$

$$= MN,$$ the product of the two given numbers.

A similar proof can be produced for two 3-digit numbers.

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Case (B): Suppose you are multiplying two 4-digit numbers N and M, both of which are slightly over 1000. The supplement of N is $(N - 1000)$, and the supplement of M is $(M - 1000)$. An expression for the first four digits of the product is $(N + (M - 1000))$ and for the last three digits is $(N - 1000)(M - 1000)$.

In expanded notation, the product would be

$$1000(N + (M - 1000)) + (N - 1000)(M - 1000)$$

$$= 1000N + 1000M - 1000000 + MN - 1000M - 1000N + 1000000$$

$$= MN, \text{ the product of the two numbers.}$$

For two 3-digit numbers the proof is similar.

HINTS TO FURTHER INVESTIGATIONS

1. Yes. Note $98^2 = 9604$ and the first two digits equal $98 - 2 = 96$ and the last two digits equal $2 \times 2 = 04$. Also, $106^2 = 11236$ and the first three digits equal $106 + 6 = 112$ and the last two digits equal $6 \times 6 = 36$.

2. First three digits: $N - (1000 - M)$

   Last three digits: $(1000 - M)(1000 - N)$

   Product $= 1000(N - (1000 - M)) + (1000 - M)(1000 - N)$

   $= 1000N - 1000000 + 1000M + 1000000 - 1000N - 1000M + MN$

   $= MN.$

3. First three digits: $N + (M - 100)$

   Last two digits: $(M - 100)(N - 100)$

   Product $= 100(N + (M - 100)) + (M - 100)(N - 100)$

   $= 100N + 100M - 10000 + MN - 100M - 100N + 10000$

   $= MN.$

4. If the digits in the two numbers are ab and ac, respectively, then the product of the two numbers is $(10a + b)(10a + c) = 100a^2 + 10ac + 10ab + bc$

   $$= 100a^2 + 10a(b + c) + bc$$

   $$= 100a^2 + 10a(10) + bc$$

   $$= 100a^2 + 100a + bc.$$

   Also, $100a(a + 1) + bc = 100a^2 + 100a + bc$.

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THE TRICK: Computing Compliments

1. Have someone suggest two 2-digit numbers. The trick works best if both numbers are slightly below 100 or slightly above 100.
2. While someone writes these two numbers in columns, you already know the product of the two numbers.

UNRAVELING THE TRICK

1. The complement of a 2-digit number, x, is $(100 - x)$. If you are multiplying two 2-digit numbers, N and M, write an algebraic expression for the difference of one of the numbers and the complement of the other.

2. Write an expression for the product of the complements of the two numbers.

3. Now, write an expression for 100 times the expression in #1 above; then, add this to the expression in #2 above. Simplify this expression.

4. Use the result in #3 to state a rule for determining the digits in the product $M \times N$.

5. The supplement of a 3-digit number, y, is $(y - 100)$. If you are multiplying two 3-digit numbers, N and M, write an expression for the sum of one of the numbers and the supplement of the other.

6. Write an expression for the product of the two supplements.

7. Now, write an expression for 100 times the expression in #5; then, add this to the expression in #6. Simplify the expression.

8. Use the result in #7 to state a rule for determining the digits in the product $N \times M$.

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QUESTIONS TO PONDER

1. If M and N are two 2-digit numbers close to 100, how many digits are there in $M \times N$? Suppose they were 3-digit numbers close to 1000?

2. Will the rule discovered in step #4 above work for any 2-digit numbers M and N?

3. Can this rule be extended to 3-digit numbers?

4. If M and N are 3-digit numbers close to 1000, how many digits are there in $N \times M$?

5. Will the rule discovered in step #8 above work for any 3-digit numbers M and N?

6. Can this rule be extended to 4-digit numbers?

FURTHER INVESTIGATIONS

1. Will the complement and supplement rules work when squaring a number, for example, 98 or 106?

2. Verify the complement rule for two 3-digit numbers.

3. Verify the supplement rule for two 3-digit numbers.

4. Verify the following "quick multiplication" of two 2-digit numbers whose ten's digits are the same and whose unit's digits have a sum of 10, for example, 47×43.

   The first two digits = the ten's digit times one plus the ten's digit

   The last two digits = the product of the unit's digits

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NUMBER 9 MORE COMPLIMENTS

This trick extends "quick" multiplication by carrying out two simultaneous multiplications and summing the products with greater speed than computing one product in the previous trick.

PERFORMING THE TRICK

1. Ask someone to write a 4-digit number on a blackboard.
2. Have someone else write a second 4-digit number below the first.
3. Announce that you will make the problem a little harder by writing a second problem on the board. Write the original number on the board and below it a fourth 4-digit number. (This one is of your choice.)
4. Ask someone with a calculator to find the products of each pair of numbers and then sum these two products.
5. Immediately, you write the answer on the board. When the calculator operator finishes, your prediction will be confirmed.

HOW IT WORKS

When you write the second problem, the top number you write will be the number originally written on the board. For the bottom number, write a number that is the difference of the second number and 9999. You can do this quickly, as if you were randomly choosing this number.

The first four digits in the answer will be the original number minus 1.

The last four digits in the answer will be the complement of the original number.

With a little practice, you will be able to write this 8-digit number as fast as you can move the chalk and, thus, beat the calculator operator in completing the computation.

AN EXAMPLE

Suppose the first subject chooses the number 5487, and the second subject chooses 7923. You would choose 9999 – 7923 = 2076 for your number.

5487	5487
× 7923	× 2076

As someone finds the two products and then sums the two products, you write down the answer.

The first four digits in the answer will be 5487 – 1 = 5486, and the last four digits will be 10000 – 5487 = 4513. So the answer is 54,864,513

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WHY IT WORKS

Let N and M be the two 4-digit numbers selected in steps #1 and #2 of the trick. Also, suppose the four digits in N are a, b, c, and d.

The first product is NM and the second product is NP, where you choose P such that $M + P = 9999$. Therefore, your answer should be

$$\begin{aligned} NM + NP &= N(M + P) \\ &= N(9999) \\ &= N(10000 - 1) \\ &= 10000N - N \end{aligned}$$

Using the digits a, b, c, and d, the difference above will appear as follows:

$$\begin{array}{rl} 10000N - N = & abcd0000 \\ & \underline{- \quad abcd} \end{array}$$

In the answer, the last four digits will be the complement of N, and the first four digits will be one less than N since you must borrow 1 from the digit d in the minuend.

HINTS TO FURTHER INVESTIGATIONS

1. (3-digit) First three digits = (original number) – 1

   Last three digits = the complement of the original number

   The rule is similar for 5-digit numbers.

2. (3-digit numbers) Suppose N has digits a, b, and c.

   The sum of the products $= NM + NP$

   $$\begin{aligned} &= N(M + P) \\ &= 999N \\ &= N(1000 - 1) \\ &= 1000N - N = abc000 - abc \end{aligned}$$

3. The answer will be

   First four digits = (original number) – 1

   Last four digits = complement of original number

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THE TRICK: More Compliments

1. Ask someone to write a 4-digit number on a blackboard.
2. Have someone else write a second 4-digit number below the first.
3. Announce that you will make the problem a little harder by writing a second problem on the board. Write the original number and below it a fourth 4-digit number (a number of your choice).
4. Ask someone with a calculator to find the products of each pair of numbers and then sum these two products.
5. Immediately, you can write the answer on the board. When the calculator operator finishes, your answer will be verified.

UNRAVELING THE TRICK

1. Choose variables to represent the three different numbers used in the trick.

2. Write an algebraic expression for the sum of the two products and factor this expression.

3. Suppose that the last number chosen in the trick had the property that the sum of it and the second number written on the board was 9999. How does this fact affect the sum in step #2 above?

4. In step #3 above, replace 9999 by (10000 – 1) and rewrite the expression for the sum of the two products. Rewrite this sum using the distributive property.

5. Suppose *a*, *b*, *c*, and *d* are the digits in the original number selected. Rewrite step #4 using the digits *a*, *b*, *c*, and *d*. From this expression, explain how you can predict the last four digits in the answer and the first four digits in the answer.

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QUESTIONS TO PONDER

1. Would the trick work for two 3-digit numbers? Two 5-digit numbers?

2. Could the first number be 4 digits and the second number 3 digits?

3. If the sum of the original number and the last number selected were 10000 instead of 9999, how would this affect your prediction rule?

4. Would the trick work if the second number were identical to the first number selected?

FURTHER INVESTIGATIONS

1. How would you predict the sum of the two products for 3-digit numbers? 5-digit numbers?

2. Verify the two rules in #1 above.

3. How is your prediction affected if the first and second numbers written down in the trick are identical?

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NUMBER 10 OLD-FASHIONED CIPHERING

You may have heard the following story about the mathematician Gauss. When his teacher tried to keep the students busy summing long sequences of positive integers, the young Gauss devised the formula for summing arithmetic sequences and, thus, was able to instantly provide the required sums.

It was common practice in the early nineteenth century for teachers to dictate long column-addition problems for their students to copy on their slate boards. When a student completed one of these sums, the teacher would look at the answer and immediately tell the student if the sum was correct. Were all these nineteenth-century teachers "lightning" calculators? No, but they had a trick like the one that follows.

PERFORMING THE TRICK

1. Hand your subject a calculator and a set of cards, each of which has twelve 5-digit numbers written in columns. (See pages 53–55 for a sample set of these cards.)
2. Ask the subject to pick out any one of the cards and sum the twelve numbers, using the calculator.
3. By looking at the card selected, you will appear to mentally sum the columns and write the answer before your subject can key the first few numbers into the calculator.

HOW IT WORKS

The sets of numbers on the sample cards all use the same "key" for determining the sum. This "key" is stated below and is illustrated using card #1 from the set of twenty-five cards included in the trick.

KEY FOR WRITING THE SUM	CARD #1	USING THE KEY ON CARD #1
FIRST DIGIT = the sum of the first and last digits in row one (if this is greater than 9, take the difference)	14836 28672 79654 93487 82853 79769 53857 86738 79306 95873 54466 17874	First digit = 1 + 6 = 7
SECOND DIGIT = the larger of the first and last digits in row one		Second digit = 6
THIRD DIGIT = the third digit in row six		Third digit = 7
FOURTH DIGIT = the second digit in row seven		Fourth digit = 3
FIFTH DIGIT = the first digit in row eight		Fifth digit = 8
SIXTH DIGIT = the first digit in row seven		Sixth digit = 5
		Thus, the sum = 767,385

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WHY IT WORKS

It just does! A great deal of effort went into constructing the sample cards so that the sum of each set of figures fits the "key." The effort pays off, however, when you display your facility at "lightning" calculation. The sum is well hidden in the twelve numbers so that no one will discover the trick, unless you tell.

HINTS TO FURTHER INVESTIGATIONS

2. The first digit in the sum is determined by finding the difference between the first and last digits in row #1. The second digit is found by taking the larger of the first and last digits in row #1. The last four digits are the sequence listed in the hint.

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THE TRICK: Old-fashioned Ciphering

1. Hand a subject a calculator and a set of cards, each of which has twelve 5-digit numbers written in columns. See the sample cards included with the trick.
2. Ask the subject to pick out any one of the cards and sum the twelve numbers, using the calculator.
3. By looking at the card selected, you will appear to mentally sum the columns of figures and then write the answer before the subject can key the first few numbers into the calculator.

UNRAVELING THE TRICK

The sums for the first four sample cards are: 767385, 658792, 762674, and 768239.

1. Compare the first digits in each sum with the first and last digits in row #1 on the corresponding card.

2. Compare the second digit in each sum with the first and last digits in row #1 of the corresponding card.

3. List the following numbers: the third digit in row #6, the second digit in row #7, and the first digit in row #8. Relate these digits to those in the corresponding sum.

4. Find a location in row #7 on the card that has a digit equal to the unit's digit in the corresponding sum.

5. Use the observations in #1 to #4 above to state a set of rules (or a "key") for writing the sum of the numbers on each card.

QUESTIONS TO PONDER

1. Does your set of rules (or "key") work on card #5? If not, how could you alter the rules to accommodate card #5?

2. Locate other cards that are similar to card #5.

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FURTHER INVESTIGATIONS

1. Create an additional card consisting of twelve 5-digit numbers that uses the "key" you developed for finding the sum of the numbers on each card.

2. Develop a key for the sums of the numbers on the five cards below. (Hint: Consider the following sequence of numbers: fourth digit in row #4, third digit in row #5, second digit in row #6, and second digit in row #5.)

#1	#2	#3	#4	#5
95685	87924	28754	87924	28754
87524	76809	99856	76809	99856
96547	93857	47267	93857	47267
95388	69755	17847	69755	17847
65878	78758	32975	78758	32975
57853	78685	18283	78685	18283

3. For the five division problems on the next page, verify that the "key" below works in each case; then construct five more such problems.

   The digits in the quotient and remainder can be determined from the three digits in the divisor as follows:

KEY FOR QUOTIENT	PROBLEM #1
first digit = $\frac{1}{2}$ (1st digit + 3rd digit)	$\frac{(3+5)}{2} = 4$
second digit = $\frac{1}{2}$ (2nd digit + 3rd digit)	$\frac{(6+5)}{2} = 5$ *
third digit = $\frac{1}{2}$ (3rd digit)	$\frac{5}{2} = 2$ *
fourth digit = $\frac{1}{2}$ (1st digit + 2nd digit)	$\frac{(3+6)}{2} = 4$ *

* omit any remainders in the division

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KEY FOR REMAINDER	PROBLEM #1
first digit = (1st digit) – 1	3 – 1 = 2
second digit = 9 – (2nd digit)	9 – 6 = 3
third digit = 9 – (3rd digit)	9 – 5 = 4

Thus, the quotient is 4524 and the remainder is 234 for problem #1.

#1 1651494 ÷ 365 =

#2 2488766 ÷ 369 =

#3 4765891 ÷ 697 =

#4 898607 ÷ 372 =

#5 5211991 ÷ 962 =

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SAMPLE CARDS FOR TRICK NUMBER 10

#1	#2	#3	#4	#5
14836	14875	16896	17286	17269
28672	83428	67839	48797	97697
79654	59215	72916	24875	81473
93487	68765	87356	92687	92855
82853	54917	62813	81796	71639
79769	35865	96271	66835	69843
53857	27381	46839	92716	76207
86738	92591	76523	35479	48796
79306	83682	38478	84528	98529
95873	75470	51343	71657	69875
54466	14856	82726	69867	91768
17874	47747	62674	81716	82696

#6	#7	#8	#9	#10
23865	23879	24363	24865	27609
45672	58364	48735	92717	35924
98816	87658	60723	58479	87614
65592	69736	52564	36874	92096
78367	56974	75846	97826	78659
85779	85726	32126	87929	68328
58637	28978	26934	71926	96580
49583	66329	65329	67548	72968
38869	95873	16795	36766	58346
45875	67954	32954	35930	49842
67874	98726	63517	86592	78651
98916	57665	31776	61715	47062

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SAMPLE CARDS FOR TRICK NUMBER 10

#11	#12	#13	#14	#15
25726	35672	34879	27609	25726
87357	18769	65710	35924	87357
74968	35668	43658	87614	74968
91831	83927	27339	92096	91831
56873	16548	88960	78659	56873
48917	35876	45932	68328	48917
65381	92834	57639	96580	65381
95736	48265	64813	72968	95736
58754	36968	95476	58346	58754
86379	76543	73968	49842	86379
89544	36528	48755	78651	89544
88130	20651	52636	47062	88130

#16	#17	#18	#19	#20
42671	37965	43872	42865	35872
32363	48376	36541	92768	61385
28672	95483	73892	95618	54973
35935	72659	62961	87634	43681
58397	54738	56875	92981	26875
65483	91863	76761	86569	18439
78679	65747	18273	38736	56837
21863	98763	36596	55875	21819
33972	72574	45843	93461	76382
52867	96716	56357	86735	53871
25275	57375	74563	85439	35953
68650	66337	65297	97172	48538

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SAMPLE CARDS FOR TRICK NUMBER 10

#21	#22	#23	#24	#25
37589	34672	29366	34865	34675
67487	53871	86725	76834	55397
79634	69264	95834	92768	66872
52031	78537	63749	76865	83951
62673	37659	84635	84938	75647
85683	23890	91527	75469	69364
24673	83972	58073	98763	58237
36875	17486	95816	85437	45872
65937	25393	43680	51866	96355
48650	16437	54879	36758	84897
86643	38398	86664	94687	98364
48557	58739	74947	45639	84214

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NUMBER 11 A TRICK FOR ALL AGES

There are dozens of tricks for predicting a person's age. Every magician knows at least one. Of course, these tricks are just variations on "think of a number" where the number happens to be the subject's age. The variation presented here is quite effective because the magician carries out the final three computations, thus keeping the trick hidden from the subject. The magician can only predict what the subject reports as being his or her age, however. If you want to know whether the subject is telling the truth, an extension of this trick is provided in *FURTHER INVESTIGATIONS*.

PERFORMING THE TRICK

Ask your subject to carry out the following computations:

1. Write your present age.
2. Add to this your age next year.
3. Multiply this sum by 5.
4. Choose any single-digit number and multiply it by 9.
5. Subtract the result in step #4 from the result found in step #3.
6. Tell me this difference.

At this point, you can predict the subject's age (or at least what the subject lists as his or her age in step #1 of the trick).

HOW IT WORKS

When the result is announced, mentally subtract 5. Then delete the unit's digit and add this deleted digit to the remaining number. This is your prediction of the subject's age.

BEWARE: This does not work if your subject's age is a single digit. (See #1 under *FURTHER INVESTIGATIONS*.)

AN EXAMPLE

Suppose your subject writes 39 as his age in step #1. These computations follow:

Step # in trick	Computation
1	39
2	$39 + 40 = 79$
3	$5 \times 79 = 395$
4	$4 \times 9 = 36$, assuming the subject chose the digit 4
5	$395 - 36 = 359$
6	The subject announces "359."

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Now, you subtract 5, giving 354. You delete the 4 and add it to 35, giving 39, the age written by the subject in step #1. Your subject should not complain since this is the age he recorded. If you want the truth, see #3 under *FURTHER INVESTIGATIONS*.

WHY IT WORKS

Let A represent the subject's age and x represent the single-digit number selected by the subject in step #4. These computations follow:

Step # in the trick	Result of computations
1	A
2	$A + (A + 1) = 2A + 1$
3	$5(2A + 1) = 10A + 5$
4	$9x$, where the subject chooses a single-digit number x
5	$10A + 5 - 9x$, which is the number announced by the subject

Here you subtract 5, giving $10A + 5 - 9x - 5 = 10A - 9x$. Before proceeding, consider the following table of values for x, $9x$, and u, representing the unit's digit of the product $9x$.

x	$9x$	u
0	0	0
1	9	9
2	18	8
3	27	7
4	36	6
5	45	5
6	54	4
7	63	3
8	72	2
9	81	1

Note that $x + u = 10$, or $x = 10 - u$, except for $x = 0$.

If $A > 9$, then the number $10A$ has a 0 in its unit's place and a value of at least a 9 in the places to the left of the units. If u represents the unit's digit in the product $9x$, then $10A - 9x$ has the digit $10 - u = x$ as its unit's digit. This is the digit that the magician deletes. Deleting this digit not only subtracts x, however, but also eliminates a place value, that is, divides the number by 10.

We have $10A - 9x$ at step #5. Deleting the unit's digit, x, gives us $\frac{10A - 9x - x}{10}$ =

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$\frac{10A - 10x}{10} = A - x$. Now adding the deleted digit, x, we have $A - x + x = A$, the subject's reported age.

HINTS TO FURTHER INVESTIGATIONS

1. Suppose the age is 8 and the single digit selected by the subject is 9. These computations follow:

   8

   $8 + 9 = 17$

   $5 \times 17 = 85$

   $9 \times 9 = 81$

   $85 - 81 = 4$

   $4 - 5 = -1$ Here the trick fails.

   Similar arguments can be devised for other ages less than 9.

2. Suppose the ten's and unit's digits of the birth year are t and u. Then we have the following computations:

   t

   $5t$

   $5t + 2$

   $10t + 4$

   $10t + 4 + u$

   $97 - (10t + 4 + u) = 93 - (10t + u)$

   This would be the subject's age in the year 1993.

3. Suppose the last two digits of the birth year are t and u. These computations follow:

   A

   $A + (A + 1) = 2A + 1$

   $10A + 5$

   $10A + 5 + u$.

   Thus, $10A + 5 + u - 5 = 10A + u$. The first two digits predict the age. Show how the last digit can be used to determine if the subject is telling the truth.

4. The result of step #2 in the trick is a multiple of 9. If you cast out nines, the remainder would be 0. Thus, when you sum the digits, S, in the announced number, the subject's age has the same remainder as the sum of the digits. Adding or subtracting nines from S, you will eventually arrive at the subject's age. You will have to estimate the subject's age, however, to know when to stop adding or subtracting nines.

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THE TRICK: A Trick for All Ages

Ask a subject to carry out the following computations:

1. Write your present age.
2. Add to this your age next year.
3. Multiply this sum by 5.
4. Choose any single-digit number and multiply it by 9.
5. Subtract the result in step #4 from the result found in step #3.
6. Tell me this difference.

At this point, you will be able to predict the subject's age.

UNRAVELING THE TRICK

1. Choose a variable to represent the subject's age.

2. Select another variable to represent the single-digit number selected in step #4 of the trick.

3. Using these variables, write expressions for the computations in each step of the trick. Note that the value of the expression in step #5 is announced by the subject.

4. Subtract 5 from the announced number and write this expression.

5. Make a 3-column table as follows: column #1, possible values for a single-digit number (call this x): column #2, nine times the value in the first column ($9x$); and column #3, the unit's digit of the numbers in the second column (call this u).

   What is the relationship between the values in columns #1 and #3? Write an equation to represent this relationship.

6. If u represents the unit's digit of the product found in step #4 of the trick, what is the unit's digit for the number in step #4 above?

7. Show that deleting this unit's digit, dividing by 10, and adding the deleted digit will give you the subject's age.

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QUESTIONS TO PONDER

1. Why does the trick fail if the subject is under ten years old?

2. Does it work for a ten year old?

3. Why must the selected number in step #4 of the trick be a single-digit number?

4. What would be the effect on the trick if the computations in steps #1 through #3 were replaced by "multiply your age by 10"?

5. Is there any way for the magician to know whether the subject is telling the truth about his or her age?

FURTHER INVESTIGATIONS

1. Prove that the trick fails if the age recorded by the subject in step #1 is a single-digit number.

2. Analyze the following trick:

   Take the ten's digit of your birth year

   Multiply this by 5

   Add 2

   Multiply this by 2

   Add the unit's digit of your birth year

   Subtract this last sum from 97

   The result is the person's age in 1993.

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3. The following extension of the age trick was devised by the magician Royal Vale Heath. Explain why it works.

   Write your age

   Add your age next year

   Multiply by 5

   Add the last digit of the year that you were born

   Announce the result

   If you subtract 5 from the announced number, you can use the first two digits to predict the subject's age, and you can use the last digit to tell if the subject is telling the truth. Devise a method for doing this.

4. The following trick is amazing in that it requires so little information from the subject. However, it requires the magician to be perceptive enough to estimate the subject's age within 10 years.

   (1) Write any 3-digit number

   (2) Multiply by 9

   (3) Add your age to this result

   (4) Tell me your answer

   To predict the subject's age, sum the digits in the number announced; call this S. The subject's age is $S + 9t$ for some integer t. Thus, if $S = 20$, then the subject's age is 2 or 11 or 20 or 29 or 38 or etc. Explain why this trick works.

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NUMBER 12 SUMMING MONTHLY SQUARES

This activity is based on the same mathematical principles used in Trick Number 4—DIVINING SUMS. It uses the natural arrangement of the counting numbers in a monthly calendar. Your subject may select any month from any calendar and outline any square of numbers in that month. From this square, the subject makes a selection of numbers, yet you will always predict the correct sum before the subject's selection is complete.

PERFORMING THE TRICK

1. Ask your subject to select any month from any calendar.
2. Have your subject outline any square of numbers in the month (2×2, 3×3, 4×4, or 5×5). You may allow blank spaces to appear in the outlined square.
3. At this point, you write your prediction and seal it in an envelope for the subject to hold.
4. Ask your subject to select numbers from the outlined square by circling them. After a number is circled, every other number in the same row and the same column must be deleted. Thus, the subject will select four numbers in a 4×4 square.
5. Now, ask your subject to sum the circled numbers.
6. When the subject compares this sum with your prediction, they will be identical.

HOW IT WORKS

Your prediction depends upon the size of the outlined square and is determined as follows:

- For a 2×2 square: the sum of the numbers found in two diagonal corners of the square
- For a 3×3 square: 3 times the number in the center of the square
- For a 4×4 square: 2 times the sum of the numbers in two diagonal corners of the square
- For a 5×5 square: 5 times the number in the center of the square

AN EXAMPLE

Suppose the following monthly calendar is selected by the subject:

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S	M	T	W	T	F	S
		1	2	3	4	5
6	7	8	9	10	11	12
13	14	15	16	17	18	19
20	21	22	23	24	25	26
27	28	29	30			

In the calendar, suppose the subject outlines the following 4 × 4 square of numbers:

	1	2	3
7	8	9	10
14	15	16	17
21	22	23	24

As soon as the square is identified, you can make your prediction:

$2 \times (21 + 3) = 2 \times 24 = 48.$

Now, suppose your subject circles the numbers shown below, always deleting the other numbers in the same row and column as the circled value.

	1	(2)	3
7	8	9	(10)
14	(15)	16	17
(21)	22	23	24

The sum of the circled numbers is 2 + 10 + 15 + 21 = 48, which matches your prediction.

WHY IT WORKS

Consider expressing each day in the form $7Q + R$, where $Q = 0$ for the first week and $Q = 1, 2, 3,$ and 4 for subsequent weeks. The R values depend upon what day of the week the selected month begins, with the starting day being 1. If the month begins on Sunday, the R values for the days of the week are 1, 2, 3, 4, 5, 6, and 7. But the calendar below begins on a Wednesday. Therefore, the R values for the seven days of the week starting with Sunday are -2, -1, 0, 1, 2, 3, and 4.

S	M	T	W	T	F	S
			1	2	3	4
5	6	7	8	9	10	11
12	13	14	15	16	17	18
19	20	21	22	23	24	25
26	27	28	29	30	31	

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The Q values for the five weeks are 0, 1, 2, 3, and 4. Thus, on the calendar above, the 19th is expressed as 7(3) + (-2), i.e., $7Q + R$, where $Q = 3$ and $R = -2$. Important to the trick is the fact that the Q and R values are sequential integers.

Suppose a 4×4 block of numbers is outlined on the monthly calendar selected by the subject. Since the subject selects four numbers of the form $7Q + R$ and no two of these are in the same row or same column, each Q value and each R value is represented exactly once in the sum. Thus, the sum computed by the subject will appear as follows:

$$S = (7Q_1 + R_1) + (7Q_2 + R_2) + (7Q_3 + R_3) + (7Q_4 + R_4) \text{ or}$$

[A] $S = 7(Q_1 + Q_2 + Q_3 + Q_4) + (R_1 + R_2 + R_3 + R_4)$, where the Qs and Rs are sequential integers.

Since the Qs and Rs are sequential, $Q_1 + Q_4 = Q_2 + Q_3$ and $R_1 + R_4 = R_2 + R_3$. Substituting these equivalences in the expression in [A] above, we have

$$S = 7(Q_1 + (Q_1 + Q_4) + Q_4) + (R_1 + (R_1 + R_4) + R_4) \text{ or}$$

$$S = (7Q_1 + R_4) + (7Q_4 + R_1) + (7Q_1 + R_4) + (7Q_4 + R_1) \text{ or}$$

[B] $S = 2((7Q_1 + R_4) + (7Q_4 + R_1))$.

The prediction number for this trick is "2 times the sum of the numbers in two diagonal corners" which is the expression in [B] above.

For a 3×3 square of numbers, the key to verifying the trick is that for the sequential values of Q and R, $Q_1 + Q_3 = 2(Q_2)$ and $R_1 + R_3 = 2(R_2)$.

Arguments similar to the one above for a 4×4 square can be developed for 2×2, 3×3, and 5×5 squares of numbers.

HINTS TO FURTHER INVESTIGATIONS

1. 2×2: the sum of the numbers found in two diagonal corners; 5×5: 5 times the number in the center
2. The prediction rules are the same as those for a seven-day calendar.
3. If you move five spaces in any direction from a given number, you will land on the same number. Thus, by counting five spaces, you will know the values of the numbers which were covered by the subject. Mentally adding these three values, you can predict the sum.

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THE TRICK: Summing Monthly Squares

1. Ask a subject to select any month from any calendar.
2. Have the subject outline any square of numbers in the month (2×2, 3×3, 4×4, or 5×5). You may allow blank spaces to appear in the outlined square.
3. At this point, you are able to make a prediction of the trick's final result.
4. Ask your subject to select numbers from the outlined square by circling them. As each number is circled, have the subject delete all numbers in the same row and in the same column as the circled number. Thus, in a 4×4 square, the subject will circle only four numbers.
5. Now, ask your subject to sum the circled numbers.
6. When the subject compares his or her sum to your prediction, they will be the same.

UNRAVELING THE TRICK

Suppose the calendar below is the one selected by the subject and the subject outlines a 4×4 square of numbers.

S	M	T	W	T	F	S
1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30	31				

1. Show that in any 4×4 square, the sum of the numbers circled by the subject always equals the sum of the four numbers along the diagonal of the chosen square.

2. Show that the sum of the two numbers at the ends of the square's diagonal always equals the sum of the other two numbers in the diagonal.

3. Use the results of #1 and #2 above to state a rule which will allow you to predict the subject's sum as soon as the square is outlined on the calendar.

4. Suppose the subject selects a 3×3 square. Compare the numbers at the ends of the square's diagonal to the number in the center of the diagonal and develop a prediction rule for a 3×3 square.

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QUESTIONS TO PONDER

1. Does it make any difference in your prediction if the first of the month is not on a Sunday?

2. Does it make any difference how many days there are in the month?

3. Does the outlined block of numbers have to be a square?

4. What effect do blank spaces in the outlined square have on the trick and your prediction?

FURTHER INVESTIGATIONS

1. Determine prediction rules for a 2×2 square and a 5×5 square.

2. Consider a calendar that has only six days in a week. Find prediction rules for the following squares: 2×2, 3×3, 4×4, and 5×5.

3. Examine the table below. Explain how the following trick works:

   Ask a subject to cover any three numbers on the table below so that you cannot see them. In just a few seconds, you are able to predict the sum of the three covered numbers.

27	12	35	20	47	27	12	35	20	47
31	10	43	19	25	31	10	43	19	25
15	26	33	40	23	15	26	33	40	23
21	14	22	37	42	21	14	22	37	42
32	28	17	41	13	32	28	17	41	13
27	12	35	20	47	27	12	35	20	47
31	10	43	19	25	31	10	43	19	25
15	26	33	40	23	15	26	33	40	23
21	14	22	37	42	21	14	22	37	42
32	28	17	41	13	32	28	17	41	13

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NUMBER 13 QUEUING FOR THE QUEEN

This trick has a long history and appears in many forms. A 1760 version reads as follows:

> Many numbers which proceed from 1 or unity, in a progression according to the natural order of numbers, being placed in a round form like a ring; to discover which of those numbers any one has thought upon.
>
> (Wingate, Edmund. *A Treatise of Common Arithmetic.* London: S. Crowder and Company, 1760. p. 372)

In the version presented here, the ring of numbers is replaced by a set of playing cards arranged in the shape of the letter *Q*. A subject thinks of a number and counts off this number starting at the tip of the *Q*s tail. Although the subject chooses the number, the count will always end at the queen.

PERFORMING THE TRICK

1. Arrange a set of playing cards in front of your subject in the form of a letter *Q*. You can use the complete deck or part of it.
2. Ask your subject to think of a number. (It must be larger than the number of cards in the tail of the *Q*.)
3. While your back is turned, have your subject count off the number selected, starting at the tip of the tail and then around the right side of the circular part of the *Q*.
4. Beginning with the card located in step #3 above, ask the subject to count starting with 1 again. This time count in a clockwise direction around the circle of the *Q*, bypassing the tail.
5. When the subject informs you that the count is complete, you can announce that the subject has reached the queen.

HOW IT WORKS

When the cards are placed on the table to form the letter *Q*, note how many cards are in the tail. The circular part of the *Q* can be as large as you wish; in fact, you can use the complete deck of cards. The key to the trick is that the number of cards in the tail of the *Q* must be used to fix the location of the queen (or one of the queens). The number of cards in the tail must be counted off clockwise along the circle starting from the place where the tail meets the circle in the *Q*. At this point, you place the queen.

Using this setup, the subject always lands on the queen. If you repeat the trick, you can alter the set up by adding or deleting cards in the tail, as long as you properly place a queen to match the length of the tail. Also, the cards can be placed face down, and you can instruct the subject to turn over the card to reveal the queen.

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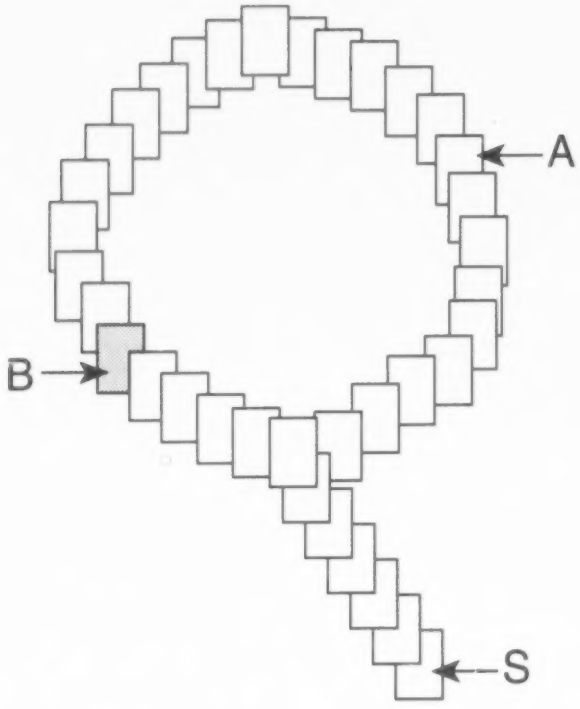
AN EXAMPLE

Suppose the cards are arranged as pictured below where the shaded card is the queen. Note that the tail contains six cards and that the queen is the sixth card counting clockwise from the tail.

Since the subject's number must be greater than six, you might ask the subject to select any 2-digit number. If the subject picks 15 and starts counting from card "S" in the diagram and then along the right side of the circle, the ending point is at card "A." Now, starting at "A" the subject counts 15 cards clockwise around the circle and ends at card "B," the queen.



WHY IT WORKS

Suppose the cards are arranged as pictured on the next page, and four specific cards (A, B, C, and D) are labelled for reference. The exact location of the other two points (E and F) will depend upon the number selected by the subject.

Suppose t represents the number of cards in the tail of the Q and n represents the number selected by the subject, where $n > t$.

When the subject starts counting n cards from location "A," the ending point will be card "E" where the number of cards from "C" to "E" is $(n - t)$. Now, from card "E" the subject counts n cards clockwise around the circle. This count can be broken down into two parts $(n - t)$ and t, since $n = (n - t) + t$. After $(n - t)$ counts, the subject is back to card "C." Counting t more cards clockwise from card "D," the ending card is at location "F" where the count from "D" to "F" is t cards. This is where you will locate the queen when setting up the trick.

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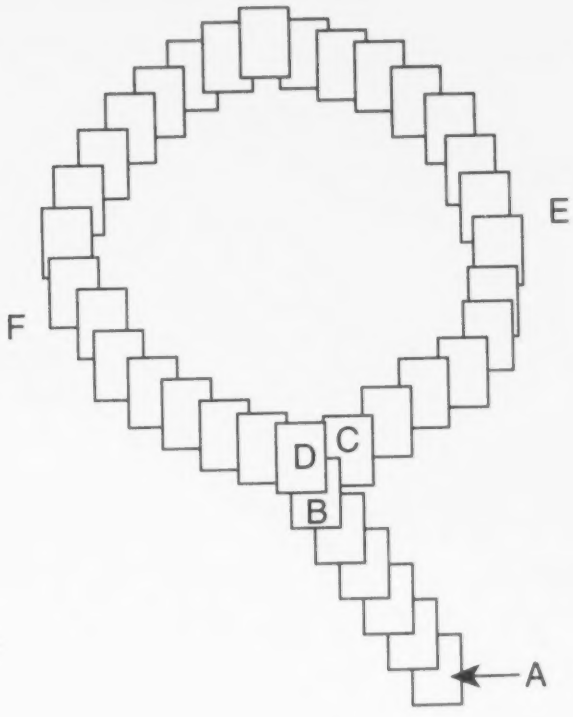

HINTS TO FURTHER INVESTIGATIONS

1. One way is to have more than one queen in the circular part of the *Q*. Then merely adding or deleting the proper number of cards in the tail will cause the subject to end at a different queen.
2. The *Q* design could easily be replaced by a "6" or a "9." Also, certain stylized lowercase letters of the alphabet could be used in place of the *Q*, suggesting another name for the trick.
3. No. You could use an arrangement of counters (like pennies). You would know at which penny the subject lands, however, and thus be able to point out that penny at the end of the trick.
4. This is a variation of the trick, PREDICTING THE HOUR, found in *Algebra Magic Tricks: Algecadabra!* (Edwards, Ronald. *Algebra Magic Tricks: Algecadabra!* Pacific Grove, CA: Critical Thinking Press & Software, 1992. p. 17)

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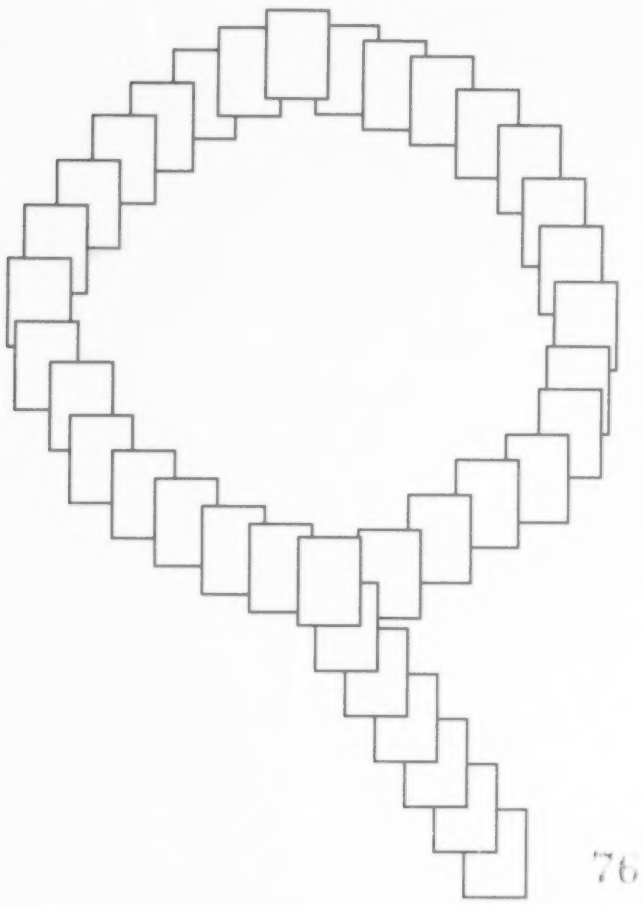
THE TRICK: Queuing for the Queen

1. Arrange a set of playing cards in front of a subject in the form of the letter *Q*. As an example, see the diagram below. The number of cards used in the *Q* is left up to you.
2. Ask the subject to think of a number. (It must be larger than the number of cards in the tail of the *Q*.)
3. While your back is turned, have the subject count off the number selected, starting at the tip of the tail of the *Q* and then around the right side of the circular part of the *Q*.
4. Beginning with the card located in step #3 above, ask the subject to count starting with 1 again. This time, have the subject count off cards in a clockwise direction around the circle, bypassing the tail of the *Q*.
5. When the subject informs you that the count is completed, you can announce that the subject has reached a queen.



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UNRAVELING THE TRICK

1. Choose a number greater than six and follow step #3 in the trick, using the sample diagram above. Note on which card you land.

2. Now, follow step #4 in the trick, and note where you land on the diagram by placing an "X" on the card.

3. Repeat the process in #1 and #2 above for different numbers greater than six. Note where you land.

4. Use your observations above to indicate where the queen should be placed in the *Q*.

5. Now that the queen is located, explain why the trick will always work.

QUESTIONS TO PONDER

1. Why must the number selected in step #2 of the trick be greater than the number of cards in the tail?

2. Could the number selected be greater than the total number of cards used to form the *Q*?

3. Could the two directions used in counting off around the circular part of the *Q* be reversed?

4. If the tail of the *Q* were longer, how would this affect your location of the queen?

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FURTHER INVESTIGATIONS

1. Show how you can vary the card the subject lands on in the trick with just minor changes in the configuration of the *Q*.

2. Some versions of this trick are referred to as the "six" or "nine" trick. Explain why this might be so. What other names might befit the trick?

3. In the trick, is it necessary to use cards or other items that are distinct from one another?

4. Analyze the following related trick:

   From a deck of cards, arrange the 13 hearts clockwise in a circle using the following sequence: A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, and K.

   Have a subject select one of the cards, unknown to you. Ask the subject to start at the "7" in the circle and silently count counterclockwise from the numerical value of the selected card to 20. At the count of 20, the subject will be pointing to the card originally selected. (For counting purposes, let A = 1, J = 11, Q = 12, and K = 13.)

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NUMBER 14 SUMMING THE UNKNOWN

There are hundreds of card tricks routinely performed by magicians. This is the only real "card trick" in this book since cards were not really needed in the last one. I chose this trick because the algebraic analysis is so nice. Also, this trick does not require great skill, sleight of hand, or special equipment—just a deck of cards and a subject. Only the subject handles the cards, which are totally out of your sight. Yet with just a few simple computations, you can make a most amazing prediction.

PERFORMING THE TRICK

Give your subject a deck of playing cards, and ask him or her to carry out the following steps while your back is turned to the subject:

1. Shuffle the deck of cards.
2. Place one card face down on the table, but note the number on the card. (Let the value of an ace be 1 and the values of the jack, queen, and king each be 10 for this trick.)
3. Start counting from the number of the face-down card up to 12, placing cards in a pile on top of each other as you count. Thus, if an "8" is face down, the subject will place 4 cards on top.
4. Place another card face down on the table, noting its number. Then repeat step #3 for this card.
5. Repeat the process in steps #2 and #3 until you run out of cards or cannot complete a pile. Keep any leftover cards separate from the completed piles.
6. When the subject indicates that the process is complete, you can turn around and announce that you are going to sum the numbers of the bottom cards in each pile. In a few seconds, you can reveal this number and have the subject turn over the piles and verify your prediction. It always works.

HOW IT WORKS

When your subject announces the process is done, you can turn around and ask the subject if there are any cards leftover. Take these cards, if any, and inconspicuously count them as you talk. Announce that you are going to sum the numbers on the bottom of each pile.

Note the number of piles and compute as follows: (1) subtract 4 from the number of piles, (2) multiply this number by 13, and (3) add to this the number of leftover cards. This result is your prediction for the sum. Now, ask your subject to verify this by turning over the piles, exposing the bottom cards, and summing them. These two sums will match.

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AN EXAMPLE

Suppose the following steps are followed by your subject.

- A 3 is turned face down, and thus 9 cards are placed on top of it.
- A 10 is turned faced down, and 2 cards are placed on top.
- An 8 is turned face down, and 4 cards are placed on top.
- An ace is turned face down (an ace is a 1), and 11 cards are placed on top.
- A 5 is turned face down, and 7 cards are placed on top.
- A king is turned face down (a king is a 10), and 2 cards are placed on top.
- A 3 is turned face down, and 9 cards are placed on top.
- Finally, an 8 is turned face down, but since there are no cards left in the deck, this pile cannot be completed. Hence, there is 1 card leftover.

As you turn and ask if there were any leftover cards, you note the single leftover card and that there are seven completed piles. Now, you compute mentally (1) $7 - 4 = 3$, (2) $3 \times 13 = 39$, and (3) $39 + 1 = 40$. This number is your prediction for the sum of the bottom numbers of the seven piles.

On your instructions, the subject turns over the seven piles, exposing these cards: 3, 10, 8, ace, 5, king, and 3. The sum of these numbers ($3 + 10 + 8 + 1 + 5 + 10 + 3 = 40$) matches your prediction.

WHY IT WORKS

If your subject turns up a card with number value x and places cards on top of this up to a count of 12, there will be $(12 - x + 1)$ or $(13 - x)$ cards in this pile.

Now, suppose there are p piles of cards when the trick is completed, with r cards leftover. Also, suppose that the values of the bottom cards in these p piles are

$$x_1, x_2, x_3, \ldots, x_p.$$

The total number of cards in the p piles is $(13 - x_1) + (13 - x_2) + \ldots + (13 - x_p)$.

If r is added to this expression, then the sum should be 52 (a complete deck of cards). Thus,

$$(13 - x_1) + (13 - x_2) + \ldots + (13 - x_p) + r = 52$$

$$(13 + 13 + \ldots + 13) - (x_1 + x_2 + \ldots + x_p) + r = 52$$

In the equation above, $x_1 + x_2 + \ldots + x_p$ is the sum of the values of the bottom cards in the p piles and 13 repeats p times, therefore

$$13p - (\text{sum of bottom numbers}) + r = 52$$

$$(\text{sum of bottom numbers}) = 13p - 52 + r$$

$$(\text{sum of bottom numbers}) = 13p - 13 \times 4 + r$$

$(\text{sum of bottom numbers}) = 13(p - 4) + r$, which is the stated rule for predicting the sum

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HINTS TO FURTHER INVESTIGATIONS

1. The sum of the bottom cards = $2(7p - 26) + r$.
2. Let d equal the number of cards handed to you and r equal the leftover cards. If x_1 and x_2 are revealed by the subject, you can predict x_3 as follows:

   $52 = d + r + (14 - x_1) + (14 - x_2) + (14 - x_3)$

   $52 = d + r + 42 - (x_1 + x_2) - x_3$

   $x_3 = d + r - 10 - (x_1 + x_2)$
3. Number of cards in deck = 52

   Number turned up = 18

   Number face down = 52 – 18 = 34

   Number of cards in subject's deck = 34

   Number turned up in subject's deck = x

   Number face down in subject's deck = $34 - x$

   Number of cards in your deck = 18

   Number turned up in your deck = $18 - x$ (Total turned up minus number turned up in subject's deck)

   Number face up in your deck = $18 - (18 - x) = x$.

   Hence, if you just turn your deck over, the number of face-up cards will match in the two piles.

See *Algebra Magic Tricks: Algecadabra!* FURTHER INVESTIGATIONS #5 on page 36.

(Edwards, Ronald. *Algebra Magic Tricks: Algecadabra!* Pacific Grove, CA: Critical Thinking Press & Software, 1992. p. 36)

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THE TRICK: Summing the Unknown

Give a deck of cards to a subject and ask him or her to carry out the following steps while you turn your back:

1. Shuffle the deck of cards.
2. Place one card face down on the table, but note the number of the card. (Let the value of an ace be 1 and the values of the jack, queen, and king each be 10 for the trick.)
3. Start counting from the number of the face-down card up to 12, placing cards in a pile on top of each other as you count. Thus, if an "8" is face down, the subject will place 4 cards on top.
4. Place another card face down on the table, noting its number. Then repeat step #3 for this card.
5. Repeat the process in steps #2 and #3 until you run out of cards or cannot complete a pile. Keep any leftover cards separate from the completed piles.
6. When the subject indicates that the process is completed, you can turn around and announce that you are going to sum the numbers of the bottom cards in each pile. In a few seconds, you can reveal this number and have the subject turn over the piles and verify your prediction. It always works.

UNRAVELING THE TRICK

1. If x is the value of the card placed face down on the table, write an algebraic expression for the number of cards in this completed pile.

2. Suppose there are five completed piles at the end of the trick and the bottom numbers of these piles are a, b, c, d, and e. Write an algebraic expression for the total number of cards in the five piles.

3. If r represents the number of leftover cards, write an equation that relates the expression in #2 above, the value of r, and the total number of cards in the deck.

4. Solve the equation from #3 above for the sum of the bottom numbers of the five piles.

5. Use your observations above to state a prediction rule for the trick.

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QUESTIONS TO PONDER

1. Can the numerical values of the ace, jack, queen, and king be changed in the trick?

2. Could the "count of 12" used in stacking the cards be changed to 13? 11?

3. How is your prediction affected if there are no cards leftover?

FURTHER INVESTIGATIONS

1. Rewrite the prediction formula under these changes in the trick:

   The ace is 1.

   The jack is 11.

   The queen is 12.

   The king is 13.

   When piling cards, the subject counts to 13.

2. Analyze the following variation of the trick outlined in #1 above:

   After you determine that there are *r* cards leftover, ask your subject to pick up and hand you all but three of the completed piles. As inconspicuously as possible, determine the number of cards handed to you; call this *d*. (You can do this by noting the bottom card of each pile.) Ask the subject to turn over any two of the remaining piles, exposing the bottom cards. At this point, you can identify the remaining (hidden) bottom card.

   How can this be done?

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3. Analyze the following card trick:

   Shuffle a deck of cards and randomly turn 18 of these cards face up in the deck.

   Hand the deck to your subject and ask the subject to shuffle the deck several times to mix the face-up and face-down cards.

   Ask your subject to hand you the first 18 cards from the shuffled deck and place the remainder of the deck on the table. Do not examine the cards given to you, and hold them behind your back.

   Announce that it is impossible for anyone to know how many of your cards are face up, but according to the laws of probability, there are probably more in the deck on the table than in yours. However, you are going to turn over some of your cards so that the number of face-up cards in your deck will exactly match the number on the table. In a few seconds, you place your deck on the table next to the subject's. Ask the subject to count how many cards are face up in each pile. They are the same.

   How do you do this?

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NUMBER 15 SO HOW AM I DOING?

Everyone likes praise. So here is a trick that allows you to shower as much (or little) praise as you wish on your subjects, whether students in a class, a group of friends, or family members. You ask each person to carry out some computations that are quite unique to each individual. Yet, each person will score as high or as low as you choose on your rating scale. In this version of the trick, everyone scores a 10!

PERFORMING THE TRICK

1. Ask each subject to write the following numbers in a column:
   - Your age as of your last birthday
   - The number of keys on your key chain (or something like that)
   - The number of letters in your last name
   - How clever you are on a scale of 1 to 10
2. Ask your subjects to sum the column of figures.
3. Now, subtract the sum of the digits of this number from the result found in step #2.
4. Sum the digits in this difference. (If the answer is not a single-digit number, sum the digits again.)
5. Square the result.
6. Subtract 1.
7. Divide by 8, and this result is your score.
8. At this time, show your subjects your rating scale. Here is a sample you might use.

Your Score	How You're Doing
0–2	mediocre
3–5	quite good
6–8	great
9	brilliant
10	a real genius

Of course, you can construct your own scale. I like the one my daughter created in her early years and used to rate everything from practicing the piano to sleeping.

[Continued on next page.]

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Your Score	How You're Doing
1	yuck
2	not so good
3	in the middle
4	kind of good
5	alright
6	okay
7	pretty good
8	rather good
9	excellent
10	great, superb

HOW IT WORKS

In this version of the trick, the result is always a "10."

WHY IT WORKS

After step #2 in the trick, the subject's number, N, could be a 1-, 2-, or 3-digit number depending on the person's age, etc. Let us suppose it is a 3-digit number with digits a, b, and c. Then, $N = 100a + 10b + c$. The computation in step #3 gives: $(100a + 10b + c) - (a + b + c) = 99a + 9b = 9(11a + b)$. Thus, the result is a multiple of 9.

If a number is divisible by 9, then so is the sum of its digits. Therefore, the result of step #4 will yield a single-digit multiple of 9, namely, 9. The last three steps in the trick produce the numbers: $9^2 = 81$, $81 - 1 = 80$, and $80 \div 8 = 10$, everyone's score.

HINTS TO FURTHER INVESTIGATIONS

1. For example:

   step #5—Add 5

   step #6—Divide by 2

   step #7—Your prediction is always "7."

2. Note that $\dfrac{(100a + 10b + c) - (100c + 10b + a)}{9}$

   $= \dfrac{(99a - 99c)}{9}$

   $= 11(a - c)$

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3. Suppose N is the number selected. After casting out nines, you are left with the remainder R_1. Note that R_1 can be found by repeatedly summing the digits of N. Suppose the digit d is deleted from N. When casting out nines from the remaining digits, you will be left with remainder R_2. Thus, R_1 should be the same remainder as that found when casting out nines in $R_2 + d$. Knowing R_1 and R_2, you can predict d as follows:

   - If $R_1 = R_2$, then a 9 was deleted
   - If $R_1 > R_2$, then d = $R_1 - R_2$
   - If $R_1 < R_2$, then d = $9 - (R_2 - R_1)$

   For example, if $N = 387$, then $R_1 = 0$. If the 8 is deleted, then $R_2 = 1$. Since $R_1 = 0$, then casting out nines in $R_2 + d$ should also give a zero remainder. Therefore, $R_2 + d$ or $1 + d$ should have a remainder of 0 when casting out nines. Thus, $d = 9 - (R_2 - R_1) = 9 - (1 - 0) = 8$.

4. Suppose the original number is $N = 1000a + 100b + 10c + d$ and the digit b is deleted. Then, $(100a + 10c + d) - (a + b + c + d) = 99a + 9c - b$. If a digit had not been deleted, this would be a multiple of 9. After casting out nines (by adding the digits), the result subtracted from the nearest multiple of 9 should tell you the deleted digit. Note that if either 0 or 9 is deleted, the answer will be 0.

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THE TRICK: So How Am I Doing?

1. Ask a subject to write the following numbers in a column:
   - Your age as of your last birthday
   - The number of keys on your key chain (or something like that)
   - The number of letters in your last name
   - How clever you are on a scale of 1 to 10
2. Ask the subject to add the column of figures.
3. Now, subtract the sum of these digits from the sum found in step #2.
4. Sum the digits in this difference. (If the answer is not a single-digit number, then sum the digits again.)
5. Square the result.
6. Subtract 1.
7. Divide by 8, and this result is your score.
8. Show your subject your rating scale. Here is a sample you can use.

Your Score	How You're Doing
0–2	mediocre
3–5	quite good
6–8	great
9	brilliant
10	a real genius

UNRAVELING THE TRICK

1. How many digits might there be in the subject's number after step #2?

2. Assume the subject's number, *N*, has three digits. Choose three variables to represent the three digits in *N*. Write *N* in expanded notation.

3. Write an expression for *N* minus the sum of the digits, and simplify the result.

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4. What will be the result of step #4 in the trick?

5. Using your observation in step #4, compute the arithmetic steps for the rest of the trick to predict the final score.

QUESTIONS TO PONDER

1. Does it make any difference what numbers the subject uses in step #1 of the trick?

2. What is special about a number minus the sum of its digits?

3. What does "casting out nines" have to do with this trick?

4. Could steps #5 through #7 be altered and still have the trick work?

FURTHER INVESTIGATIONS

1. Alter the steps in the trick so that the final answer is a number other than "10."

2. Verify that if the difference of any 3-digit number and the three digits in reverse order is divided by 9, then the quotient will always be a multiple of 11.

3. Analyze the following trick:

   (1) Write a number with 3 or more digits.

   (2) Divide by 9 and tell me the remainder, R_1.

   (3) Delete any non-zero digit from the original number.

   (4) Divide the resulting number by 9 and tell me the remainder, R_2.

   (5) I can now predict the deleted digit.

   How does this trick work?

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4. Analyze the following trick:

   (1) Write down a 4-digit number.

   (2) Write down the sum of the four digits.

   (3) Delete a digit in the original number.

   (4) Write the 3 remaining digits above the sum found in step #2.

   (5) Subtract the numbers and announce the result.

   (6) Your prediction of the deleted digit is found by adding the digits in the announced number and subtracting this from the nearest multiple of 9.

   Does this always work?

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